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Game Theory

Game theory is one of the most interesting topics of discrete mathematics. The principal theorem of game theory is sublime and wonderful. We will merely assume this theorem and use it to achieve some of our early insights. To appreciate the theorem it is not necessary to know the proof. Do not let any math pedant tell you otherwise.[1] By a *game* mathematics refers to a conflict between individuals (or entities) with conflicting goals. The element of fun is not addressed in the mathematical view of games. To mathematicians nuclear war is a game and it has been studied as such.

		Bebe		
		b1	b2	b3
	a1	3	1	-2
Abe	a2	1	2	4
	a3	-3	1	5

Game 1 A Two-Person Zero-Sum Game

An example of a game is given in the box at right, **Game 1** . All of the games we study will be two-person games. In this example, the two players are Abe and Bebe. Abe has three possible moves: a1, a2, a3; and Bebe's possible moves are b1, b2, and b3. Since each player has 3 moves, this game is said to be 3-by-3 game. The *pay-off* of a move is the corresponding table

[1]Here I repeat remarks in the note preceding Chapter 15. That note also tells you where to find proofs.

entry (the table is sometimes referred to as the *pay-off matrix* of the game). For example, if Abe were to make move a1 and Bebe were to make move b3, then the pay-off to Abe would be the amount in the a1-b3 entry of the table, -2. This would be a 2 point loss for Abe and a 2 point gain for Bebe. This last property is why the game is called a *zero-sum* game: Abe's loss is Bebe's gain and vice versa. It follows that if the game is not to be trivial then each player should make his or her move without knowledge of the other player's move. For example, in this instance, if Abe knows that Bebe will do move b3, then he will make move a3. If he knows that Bebe will make move b2, then he will make move a2. Similarly, if Abe knows that Bebe will make move b1, in this instance, he will make move a1. Clearly **Game 1** is written from Abe's point of view. Another convention in this book is to add the same constant to each entry of the game so that there are no negative values. This is simply done for convenience. It has no effect whatsoever on the strategies. Its effect on the game is to add the same constant to the value of the game (the meaning of which will be explained later). For example, **Game 2** is precisely the same as **Game 1** except that each entry of the pay-off matrix has been increased by 3 points. Since the value of **Game 1** is 1.66, the value of **Game 2** is 4.66.

		Bebe		
		b1	b2	b3
Abe	a1	6	4	1
	a2	4	5	7
	a3	0	4	8

Game 2 The Previous Game Plus 3 Points

Strategies

In **Game 2** , a *strategy* for Abe means a probability vector $P = (p_1, p_2, p_3)$. Since P is a probability vector, this means (just to remind you) that each $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$. p_1 is the probability that Abe will play strategy a1; p_2 is the probability that Abe will play strategy a2; p_3 is the probability that Abe will play strategy a3. If Abe has the strategy P = (0, 1, 0), he will always play move a2. Such a strategy is called a *pure strategy*. By contrast, the strategy P = (¼, ½, ¼) is called a *mixed strategy* and must be *randomized*. This means that at any given move, Abe has probability ½ of playing move a2 regardless of his last move. Note that when we examine **Game 2** , I have not specified whether the game is a one-time game, or a game that will be played many times in sequence. The analysis done here seems to imply the latter case. However, there is no such intention. It can be argued that analysis is predicated on the assumption of a sequential series of games and that this analysis does not apply to a one time game. This is an interesting philosophical argument and the reader is encouraged to think about it. One argument goes that even if a particular game is to be played once, perhaps there will be other occasions where "equivalent" games will be played, and therefore the best approach is to play each game as if it is part of a sequence. I do not suggest that you must accept this argument.

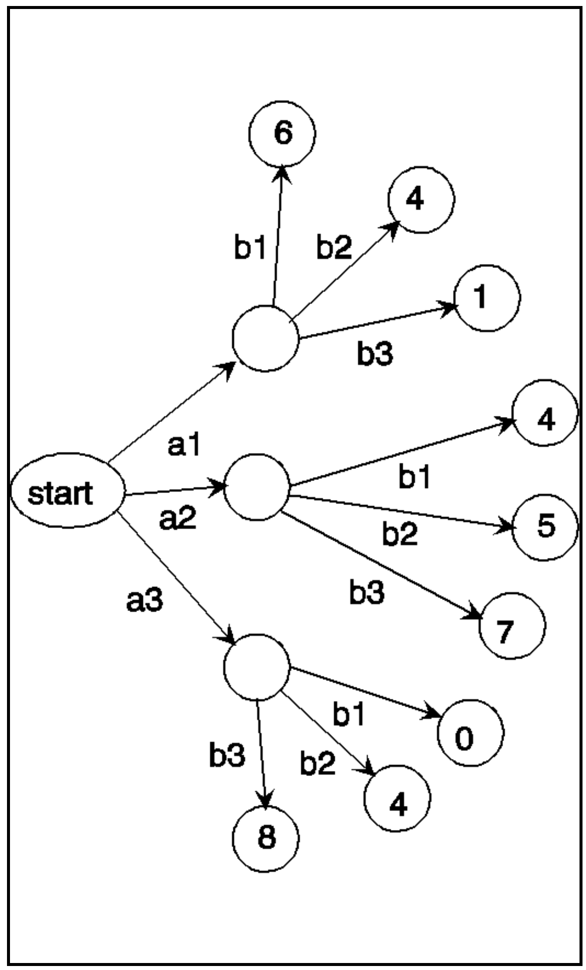


Figure 1 The Game Tree for **Game 2**

We can associate a game tree with a game such as **Game 2** . This game tree is shown in **Figure 1**. Here Abe's strategy is P = (p_1, p_2, p_3) and Bebe's strategy is Q = (q_1, q_2, q_3). The numbers on the leaves of the tree are the outcomes. It follows that the expected outcome of the game is:[1]

$$p_1q_1 6 + p_1q_2 4 + p_1q_3 1 + p_2q_1 4 + p_2q_2 5 + p_2q_3 7 + p_3q_1 0 + p_3q_2 4 + p_3q_3 8$$

The Fundamental Theorem of Game Theory

Again, we are studying two-person zero-sum games. In this environment, there is a remarkable theorem: **Each game has a definite value. There is a value such that either player can guarantee averaging that well.** For instance if the value of the game is, say 4, then Abe has a strategy such that the average outcome will be *at least* 4. Bebe will have a strategy such that the average outcome will be *at most* 4. These strategies are called *optimal* strategies. It follows that if each player uses his optimal strategy then the average outcome will be fixed at the value of the game.

Let us reconsider **Game 2** . Without knowing anything about how to find the value of the game we can find a lower and higher bound for it. We do this by consider pessimistic strategies for each player. We ask: *What is the worst that can happen for a given pure strategy?* From Abe's point of view if he plays strategy a1 the worst that can happen is that Bebe plays strategy b3 and the outcome is 1. If he plays strategy a2 the worst outcome is

[1]The same expression for the expected outcome value of the game can be more economically expressed using matrix notation as follows: $(p_1, p_2, p_3)\begin{pmatrix} 6 & 4 & 1 \\ 4 & 5 & 7 \\ 0 & 4 & 8 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$. This is an excellent example of the utility of matrices.

4 points and if he plays a3 the worst outcome is 0 points. The best of these is playing strategy a2 which guarantees that he achieves at least 4 points. This is the *maxi-min* strategy (it maximizes the minimum of the rows). The maxi-min strategy for Abe is to play a2 and assure himself of at least 4 points. For Bebe, she must look at things from the opposite perspective. She wants the minimum number of points since Abe's win is her loss. By making all of the scores positive we have Bebe losing from 0 to 8 points depending on the strategies played, but that is unimportant. The important consideration is that Bebe wants to minimize the payoff (outcome) to Abe. Looking at the game from a pessimistic point of view, if she plays strategy b1, then the worst that can happen to her is that Abe plays a1 and wins 6 points. The worst that can happen if she plays b2 is that Abe plays a2 and wins 5 points. The worst that can happen if she plays b3 is that Abe plays a3 and wins 8 points. The best of these outcomes from her point of view is when she plays b2 and Abe makes at most 5 points. By playing b2 she assures herself of losing no more than 5 points. Her pessimistic strategy is to minimize the maximum outcome of the columns. She plays a *mini-max* strategy. In fact the custom is frequently to describe both strategies as *mini-max* strategies. In principle both players are using the same approach: **pick the pure strategy that *minimizes the worst that can happen***.

		Bebe		
		b1	b2	b3
	a1	6	4	1
Abe	a2	4	5	7
	a3	0	4	8

Game 3 Same Game With Pessimistic Strategies as Indicated

Saddlepoints

In the above analysis Abe's mini-max strategy assures him of at least 4 points and Bebe's strategy assures her of losing no more than 5 points. Hence the *optimal* value, the value of the game, is between 4 and 5 points inclusive, and the optimal strategies for both players is a mixed strategy. Note that when both Abe and Bebe play the mini-max strategies the outcome is 5 points and Abe may well be satisfied.

> Only when the mini-max strategies assure the same value are they optimal. In this case the optimal outcome is a minimal value on its row and a maximum value of its column. It is called a *saddlepoint* and is the optimal value of the game. The optimal strategies for both sides are to play for the saddlepoint. That is each player has an optimal pure strategy. A saddlepoint is not necessarily unique, but its value is unique.

In **Game 2** Abe's mini-max strategy assures him of at least 4 points; Bebe's mini-max assures her of losing no more than 5 points. Since these numbers are **not** the same, this game does not have a saddlepoint and the optimal strategies for both players are mixed and the value of the game is between 4 and 5 points.

It so happens that the value of that game is 4.66. Bebe can assure herself of doing no worse than losing (on average) 4.66 points a game if she plays the strategy Q = (1/3, 2/3, 0). In that case if Abe plays any mixture at all of his first two strategies, a1 and a2, the average outcome will be 4.66. Should he ever play strategy a3 Bebe can average less then 4.66 (which is a change in her favor). For example should Abe always play a3, Bebe would average 2.66 a game with her *optimal* strategy (Q = (1/3, 2/3, 0)). However, should Abe be stupid enough to always play strategy a3, Bebe could always play strategy b1 and average 0 points a game. Coincidentally, Abe's optimal strategy is also P = (1/3, 2/3, 0). It turns out that for any mixture of strategies b1 and b2 that Bebe plays (when Abe plays his optimal strategy) she will average a loss of 4.66 points a game.

However, should Bebe ever play her strategy b3 she will lose more than 4.66 (on average).

☐ **Exercise 1** Which of the following three games have saddlepoints? What are the value(s) of those games and what are (is) the optimal strategie(s). Of the game(s) without saddlepoints find an upper bound and a lower bound for the value of the game (hint: find each player's mini-max strategy).

Abe (rows) vs. Bebe (columns)

9	1	0
5	2	4
3	1	0

Abe (rows) vs. Bebe (columns)

2	3	4
1	4	5
6	5	4

Abe (rows) vs. Bebe (columns)

3	5	2
0	5	1
0	9	0

Three Games for Homework

Dominant Strategies

Consider **Game 4**. Abe has four strategies and Bebe has three strategies. The mini-max strategy for Abe is to play a2 or a3 and assure himself of at least 5 points. The mini-max strategy for Bebe is to play b2 and assure herself of losing no more than 6 points. Since these values are different, there is no saddlepoint; the value of the game is between 5 and 6; the optimal strategies for both players

Abe \ Bebe	b1	b2	b3
a1	0	5	9
a2	9	5	5
a3	5	5	5
a4	0	6	9

Game 4 A Four by Three Game

Abe \ Bebe	b1	b2
a2	9	5
a4	0	6

Game 5 The Previous Game Reduced

will be mixed (and we still do not know how to calculate mixed strategies). However, it is possible for us to simplify the game by using ordinary logic and common sense. Frequently strategies can be eliminated from use. In this case, Abe has no reason to ever play strategy a3. The reason is that whatever Bebe does, a2 will always perform as well as a3. We say the strategy a2 *dominates* the strategy a3. Similarly, strategy a4 dominates strategy a1. From Bebe's point of view, strategy b2 dominates b3. All together we should eliminate from consideration strategies a1, a3, and b3. This leaves us **Game 5** . It is important to remember that once strategies have been eliminated, that strategies that were not dominated or dominating may change status and become dominated or dominating. Then it is possible to eliminate even more strategies.

☐ **Exercise 2** Solve the games given below. By *solve* I mean find the value for the game and find optimal strategies for each player. (Hint: since we have not studied mixed strategies your only hope is to find saddlepoints and/or to eliminate dominated strategies. If however, the last problem say, seems to come down to mixed strategies, there should be an easy way to see what the correct answer is.)

Bebe

	2	4	2
Abe	2	3	2
	3	3	2

Bebe

	0	2	1
Abe	3	2	6
	2	1	7

Bebe

	8	1	0	0
Abe	8	2	3	9
	9	4	3	8
	8	0	1	0

Bebe

	3	4	0	0
Abe	4	3	0	0
	5	5	5	0
	5	5	0	5

Four More Games for Homework

The Calculation of Mixed Strategies

We will concentrate on evaluating 2-by-2, 2-by-n, and n-by-2 games. More general games will be treated in an appendix to this chapter. An important fact about 2-by-2 games, is that a game has a saddlepoint if and only if it has dominant strategies. This rule does not hold for larger games.

When examining a 2-by-2 game, just as for any game, we look for a saddlepoint. The following analysis applies only if there is not a saddlepoint. Consider the generic 2-by-2 game **without** a saddlepoint: **Game 6** . We want a strategy (p, 1−p) for Abe such that the expected outcome of the game is unaffected by Bebe's strategy. In particular the expected outcome would be the same if Bebe played only b1 or only b2. In the first case the expected outcome is pA + (1−p)C in the second case the expected outcome is pB + (1−p)D. Setting these equal to one another we get pA + (1−p)C = pB + (1−p)D. Solving for p we get:

		Bebe	
		b1	b2
Abe	a1	A	B
	a2	C	D

Game 6 Generic 2-by-2 Game

$$p = \frac{D-C}{A-B+D-C}$$

Similarly Bebe should play strategy b1 q of the time where

nI

$$q = \frac{D-B}{A-C+D-B}$$

either case the calculation will give you a probability between 0 and 1 exclusive (a probability of 0 or 1 would be a pure strategy). If your calculation is outside of that range you failed to observe that there was a saddlepoint.

<u>Example</u>

In **Game 7** Abe's mini-max strategy is a1 since its minimum is 10 versus the minimum of 0 in strategy a2. Bebe's mini-max strategy is b1 since its maximum is 100 versus a maximum of 1000 for b2. Abe can assure himself of always doing at least 10 by strategy a1 and Bebe can assure herself of no worse than 100 by strategy b1. If these two values were equal we would have a saddlepoint. As it is, the optimal strategies are mixed strategies and the value of the game must lie between 10 and 100. To implement the above strategy subtract the columns from one another and subtract the rows from one another. It does not matter in what order you subtract one column from another, just as long as you are consistent.

		Bebe b1	Bebe b2
Abe	a1	100	10
	a2	0	1000

Game 7 A 2-by-2 Game

			Bebe b1	Bebe b2
			100	-990
Abe	a1	90	100	10
	a2	-1000	0	1000

Figure 2 Start of Solution of **Game 7**

			Bebe b1	Bebe b2
			990/1090	100/1090
Abe	a1	1000/1090	100	10
	a2	90/1090	0	1000

Figure 3 Solution of **Game 7** Continued

In **Figure 2** I have subtracted the first column from the second and the first row from the third. When this is done we get a positive answer and a negative answer, otherwise the game has a dominant strategy and a saddlepoint. The next step is to take the absolute values of the new entries and to switch them. Lastly, to get the optimal strategies, we divide each entry by the sum of the entries. This is shown in **Figure 3**. For Abe, the optimal strategy is to play a1 100/109 of the time and to play a2 9/109 of the time. For Bebe the optimal strategy is to play b1 99/109 of the time and to play b2 10/109 of the time. If either player plays his or her optimal strategy the average outcome will be the value of the game. Remember that in larger games such as 2-by-3 either player can always assure doing as well as the value of the game by playing his or her optimal strategy. However, if the other player plays badly, it is possible to do better than the value of the game. In a 2-by-2 game, the outcome is **strictly** the value of the game as long as either player plays his or her optimal strategy. To calculate the value of the game, let Abe play his optimal strategy, and assume the Bebe always plays b1. Then the expected outcome is:

$$\frac{99}{109}\cdot 0 + \frac{10}{109}\cdot 1000 = 91.743$$

and the value of the game is thus 91.743. Similarly, we could calculate the value of the game by assuming that Abe always plays his strategy a2 but Bebe plays her optimal strategy. Then the expected outcome is:

$$\frac{100}{109}\cdot 100 + \frac{9}{109}\cdot 0 = 91.743$$

Similarly, the other two analogous computations (Abe plays his optimal strategy and Bebe plays b2 or Bebe plays her optimal strategy and Abe plays a2) will yield exactly the same result.[1]

	b1	b2
a1	10	20
a2	40	0

	b1	b2
a1	10	20
a2	40	19

	b1	b2
a1	10	0
a2	0	100

Three Games for Homework

☐ **Exercise 3** Solve for the values and optimal strategies of the following games.

n-by-2 and 2-by-n Games

We will only consider n-by-2 games because 2-by-n games are exactly analogous. **Game 8** is a 4-by-2 game. We can immediately reduce it to a 3-by-2 game since Abe's strategy a2 dominates his strategy a4. Looking at the 3-by-2 game left by disregarding strategy a4, we can find no dominated strategies for either player. Abe's mini-max strategy is to play a2 and assure himself of at least 3 points. Bebe can play b1 or b2 and assure herself of losing no more than 4 points. The value of the game is between 3 and 4 points. It so happens that Abe will have an optimal strategy that

		Bebe b1	Bebe b2
Abe	a1	4	0
	a2	3	3
	a3	0	4
	a4	1	2

Game 8 A 4-by-2 Game

[1]You might object that if either player plays a pure strategy that the other player can do better than the value of the game, by playing his or her appropriate pure strategy. This is true, but when do you have an assurance, that the other player will continue to play a pure strategy? Here we get into the psychology of gamesmanship (or gamespersonship). The point is, if either player likes the value of the game, he or she can guarantee achieving it *regardless* of what the other does.

consists of a mixture of 2 of the pure strategies. This is always true for an n-by-2 game (without a saddlepoint). It is proven by the graphical analysis of n-by-2 games that will occur in the next section. For the time being we can find Abe's optimal strategy by looking at the various 2-by-2 games that are sub-games of **Game 8** .[1] We have three games to consider: they are the a1-a2 game, the a1-a3 game, and the a2-a3 game. In the a1-a2 game Bebe has a dominating strategy of b1, and Abe's strategy is a2, and the value is 3 (which is a saddlepoint). Clearly the a2-a3 game is almost identical and has a value of 3. The a1-a3 has an optimal strategy for Abe of (.5, .5) and a value of 2. Therefore Abe can consider either the a1-a2 or the a2-a3 game as optimal, and in either case he plays the pure strategy a2, and it doesn't matter what Bebe does.

☐ **Exercise 4** Find the optimal strategies and the value for **Game 9** .

		Bebe			
		b1	b2	b3	b3
Abe	a1	0	5	6	4
	a2	4	6	1	3

Game 9 A 2-by-4 Game for Homework

[1]The meaning of *sub-game* should be self-evident. It is a game derived from the first game by eliminating certain strategies from consideration. Anytime we eliminate a dominated strategy we are going to a sub-game. In this context we will eliminate a strategy that is not dominated by another single strategy.

The Graphical Analysis of Games

Graphical analysis of games is usually done to understand 2-by-n and n-by-2 games, but to understand the graphical analysis it is best to study 2-by-2 games.[1] **Game 10** is an ordinary 2-by-2 game with value 16/7. Abe's optimal strategy is (.571, .429) and Bebe's optimal strategy is (.429, .571).

		Bebe	
		b1	b2
Abe	a1	4	1
	a2	0	4

Game 10 The Value is 16/7

[1] I have seen other texts at this level present the graph for an n-by-2 game as if the interpretation is obvious. It is not obvious at all. That is why I am starting at the 2-by-2 level. I don't claim that it is now easy. Take your time and study the example. Continue only when you feel comfortable.

The graph of **Game 10** is given in **Figure 4**. There are two diagonal axes corresponding to Bebe's 2 strategies. The main trick to this type of graph is understanding the bottom axis. The numbers on the bottom axis range from 0 to 1. They represent the probability that Abe plays strategy a1. For example, if Abe has the mixed strategy (.25, .75) corresponding to playing a1 25% of the time and playing a2 75% of the time, this strategy is denoted by .25 along the base axis. The case where he plays a1 0% of the time, strategy (0, 1) is the case where he plays a2 100% of the time, and that is why the first vertical axis is denoted by *a2*. If you draw a vertical line from the point on the bottom axis denoted .25 it crosses the b1-line at the height = 1 which is what the strategy b1 would average against Abe's strategy (.25, .75). The vertical line (at .25) crosses the b2-line at 3.25 which is what Abe's strategy (.25, .75) averages against b2. Clearly, Bebe does best against (.25, .75) using b1 rather than b2 or some mix of b1 and b2. According to the graph, if Abe plays a1 less than .571 (4/7) of the time, Bebe does best with b1 (since Bebe wants the minimum outcome). If Abe plays a1 more than .571 of the time Bebe does best with b2. When Abe plays a1 .571 of the time, it doesn't matter what Bebe does: (.571, .429) is Abe's optimal strategy.

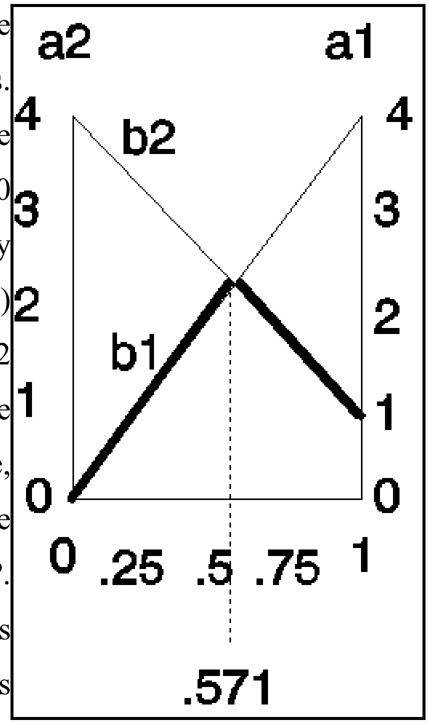


Figure 4 The Graph of **Game 10**

☐ **Exercise 5** Considering the graph of **Game 10** , if Abe plays strategy (.25, .75), and if Bebe plays a mixture of b1 and b2, where on the graph is the average outcome? (Hint: First look at the average outcomes if Bebe plays only b1 and and then if Bebe plays only b2).

☐ **Exercise 6** Graph a 2-by-2 game where there is a saddlepoint. What happens?

We will study a 3-by-2 game, because it is representative of all 2-by-n and n-by-2 games. In **Game 11** there are no dominated strategies for either player. Abe's mini-max strategy is to play a2 and to assure himself at least 2 points. Bebe's mini-max strategy is to play either b1 or b2; she will lose at most 4 points either way. Therefore there is no saddlepoint and the value of the game is between 2 and 4 points. To graph a game where Abe has three strategies to Bebe's 2 strategies, it is best to graph the game from Bebe's point of view.

		Bebe	
		b1	b2
	a1	0	4
Abe	a2	2	3
	a3	4	1

Game 11 A 3-by-2 Game

In the graph, the two vertical axes correspond to Bebe's strategies b1 and b2. The bottom axis corresponds to Bebe's strategy (p, 1−p). At the left end Bebe is playing (0, 1) (strategy b2) and the axis is labeled b2. At the right end, Bebe is playing (1, 0) and the axis is labeled b1. If Bebe plays any strategy between (0, 1) and (⅓, ⅔) Abe's best response, shown by the heavy black line, is to play a1 which averages better than a2 or a3. (Note, my graph is not perfectly scaled.) Between (⅓, ⅔) and (½, ½) Abe will play strategy a2. Between (½, ½) and (0, 1) Abe will play a3. It is precisely at (½, ½) that Abe's strategies a3 and a2 perform the same with an expected outcome of 2.5. The graph indicates that this is the lowest value that Bebe can guarantee. Abe can always play a strategy that achieves the level indicated by the extra thick line segments. This is lowest (at 2.5) when Bebe plays (½, ½). The graph indicates that if Abe plays strategy a1 when Bebe plays (½, ½) that Abe does worse than with the other strategies. From Abe's point of view he should stick to a2 and a3. Analyzing the game as a two strategy game it is clear that Abe's optimal strategy is to play a2 ¾ of the time and to play a3 ¼ of the time.

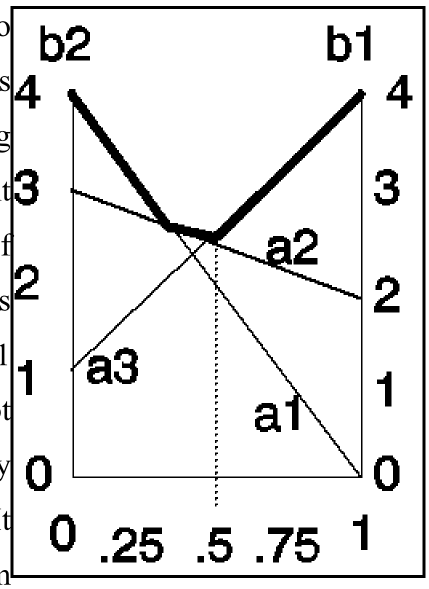


Figure 5 The Graph of **Game 11**

☐ **Exercise 7** Analyze **Game 11** by considering it as 3 different 2-by-2 games that Abe has a choice of playing. Verify that the value and optimal strategies are the same as given above.

☐ **Exercise 8** Solve **Game 8** and **Game 9** using the graphical technique.

Appendix to Chapter 17

General Solution of Two-Person Zero-Sum Games

The reason for making this section an appendix is that it relies on linear programming. Whereas many people do consider linear programming a part of discrete mathematics, it is a subject unto itself and should be studied as such.[1] I will demonstrate the general technique of zero-sum games by doing an example: **Game 1** . This game is given again as **Game 12** . Note that this game has neither dominated strategies nor a saddlepoint. We are going to find an optimal strategy (p, q, r) for Abe. Such a strategy must have the quality that it achieves at least the value of the game regardless of the strategy that Bebe plays. Let us denote the still unknown value as V. We have 4 very basic linear constraints:

		Bebe b1	Bebe b2	Bebe b3
Abe	a1	3	1	-2
	a2	1	2	4
	a3	-3	1	5

Game 12 **Game 1** again.

[1]Excellent texts on linear programming that also discuss the application to game theory are:

Linear Programming. Vašek Chvátal. W. H. Freeman. New York, N.Y. 1983.

An Introduction to Linear Programming and Game Theory, 2nd. ed. Paul R. Thie. Wiley. New York, N.Y. 1988.

- $0 \le p$
- $0 \le q$
- $0 \le r$
- $p + q + r = 1$

To these we add three more constraints that fulfill the condition given above: the average outcome of Abe's strategy against any of Bebe's strategies must be at least V.

- $3p + 1q + (-3)r \ge V$ (the average outcome against b1 must exceed V)
- $1p + 2q + 1r \ge V$ (the average outcome against b2 must exceed V)
- $(-2)p + 4q + 5r \ge V$ (the average outcome against b3 must exceed V)

Altogether we have 7 linear constraints to which we add the linear objective:

- Maximize V

In the usual simplex solution of the linear programming problem the first three non-negativity constraints will be implicit. However, it is important that V not be constrained to non-negativity. Hence the simplex formulation might be to write V as the difference of two non-negative variable: $V = V_1 - V_2$. Also, in the usual formulation, V will be moved to the left side of the constraints that it appears in. In any case linear programming problems can be solved using inexpensive software available for today's microcomputers. In fact problems of this size can be easily solved using the *optimization* item in some spread-sheets.

Solving the above game using a linear programming package, we get the value of the game, V, is 1.66. The optimal strategy for Abe is $p = \frac{1}{3}$ and $q = \frac{2}{3}$ with $r = 0$. Note that $r = 0$ despite the fact that strategy a3 is not dominated by either a1 or a2. To find Bebe's optimal strategy you can either use the same approach just used for Abe's strategy or you can use the dual variable solutions given for the solution to the above formulation. In this case it so happens that Bebe's optimal strategy looks just like Abe's: $(\frac{1}{3}, \frac{2}{3}, 0)$.

1. The first game has a saddlepoint and the value is 2. The optimal strategy for Abe is a2 and the optimal strategy for Bebe is b2.

   The second game has no saddlepoint. Abe's mini-max strategy (the row with the maximum minimum) is a3, with a minimum of 4 points. Bebe's mini-max strategy (the column with the minimum maximum) is b2 or b3 with a maximum of 5. The true value of the game must be in between 4 and 5 points.

   The third game has a saddlepoint with value 2. Abe's optimal strategy is a1 and Bebe's optimal strategy is b3.

2. In the first game, Bebe's strategy b3 dominates the other two strategies. Since b3 fixes the value of the game at 2, it does not matter what Abe does. In fact all three boxes in the rightmost column are saddlepoints.

   In game 2 Abe's strategies a2 and a3 dominate a1. Having removed a1, Bebe's strategy b2 dominates b1 and b3. At this point Abe will choose a2 and the value of the fame is 2. This again happens to be a saddlepoint.

   In game 3 a1 and a4 are dominated. Once they are removed, b1 and b4 are dominated. After b1 and b4 are removed as well, a3 dominates a2. This leaves Bebe to choose b3. The optimal strategies are thus a3 and b3, with a value of 3, which happens to be a saddlepoint.

   In game 4 strategies a1 and a2 are dominated. After they are removed, strategies b1 and b2 are dominated. This leaves a 2-by-2 game that has no saddlepoint and therefore requires mixed strategies. Since the game is totally symmetric, the optimal strategies are for Abe to play a3 and a4 each half of the time, and for Bebe to play b3 and b4 each half of the time. The value of the game is thus 2.5.

3. Game 1: Abe's strategy is (.8, .2); Bebe's is (.4, .6). The value is 16.

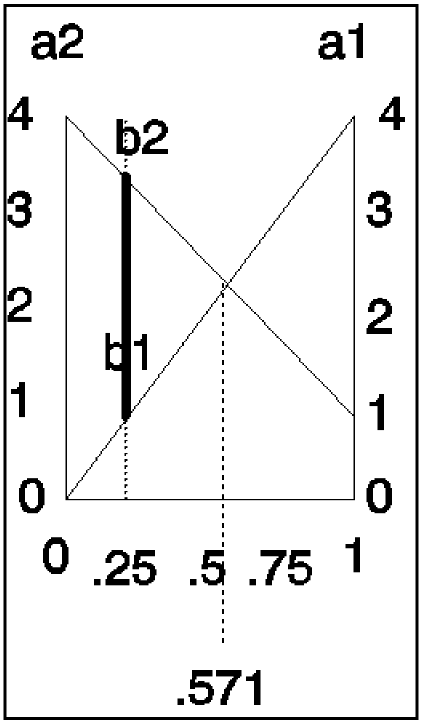
   Game 2: Abe's strategy is (21/31, 10/31); Bebe's is (1/31, 30/31). The value is 610/31.

   Game 3: Abe's strategy is (10/11, 1/11); Bebe's is (10/11, 1/11). The value is 100/11.

4. Strategy b2 is dominated. Looking at the 3 games we get:
   b1-b3 game: value is 8/3
   b1-b4 game: value is 3.2
   b3-b4 game: value is 4 (which is a saddlepoint).
   Since Bebe wants the lowest value, Bebe will play b1-b3 with strategy (5/9, 4/9) or in

terms of the original game the strategy is (5/9, 0, 4/9, 0). Abe's strategy is (⅓,⅔).

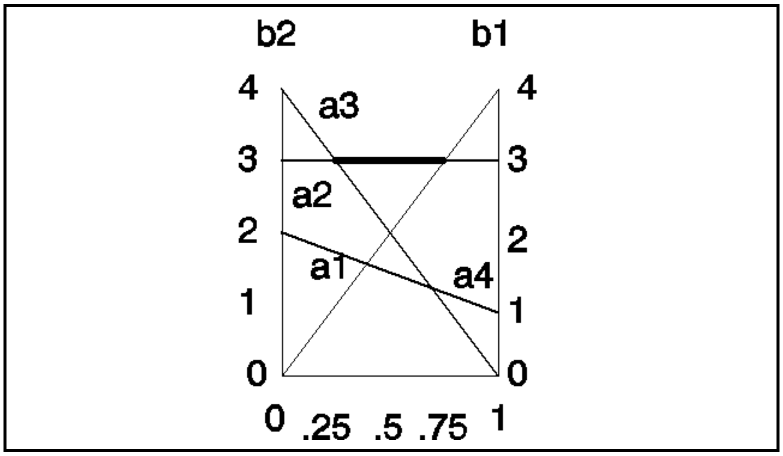
5. Were Bebe to play b1 then the outcome is at the intersection of the vertical line at .25 and the line marked b1. Similarly, were Bebe to play b2 then the outcome is at the intersection of the vertical line at .25 and the line marked b2. For a mixed strategy of b1 and b2 the outcome is on the vertical segment above .25 that connects these two points.



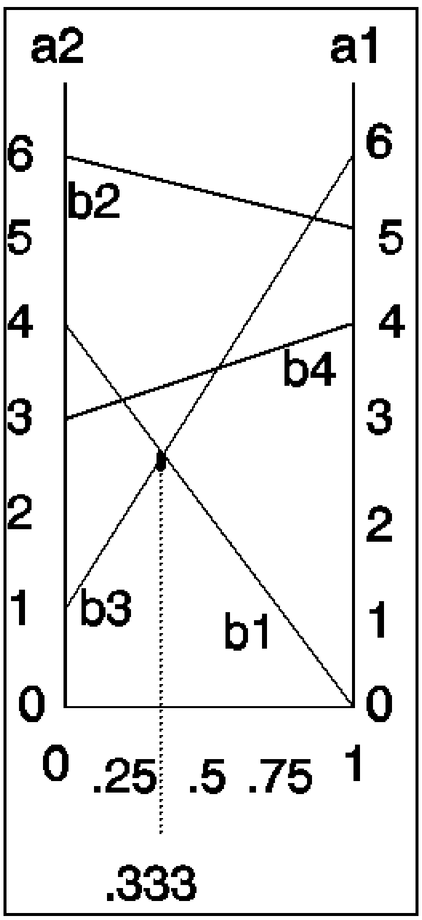
6. For a 2-by-2 game, there is a saddlepoint if and only if someone has a dominating strategy. When that strategy is graphed versus the other strategy it will be above or below the other strategy (depending on whether it is Abe's or Bebe's).

7. If you get a different answer than in the text, then one of us has made a mistake.

8.



Graph of **Game 8**



Graph of **Game 9**