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Random Processes

We are going to look at a random process that typifies what most of us think of as *random*. This process has the added virtues that it is easy to work with and it is used a great deal in mathematical modeling. In fact, it may be used more than any other process of its type. In particular what I am talking about is the *Poisson*[1] process which is described by both the Poisson distribution and the exponential distribution. (Each defines a different aspect of the process.)

Basic Assumptions

There are plenty of random processes other than the Poisson. Given below will be assumptions that apply to the Poisson process. First let us look at the general nature of a Poisson process.

We are dealing with a procession of *events*. These typically might be customers coming into a hamburger joint, or phone calls coming into an exchange, or lightning strikes hitting a lightning rod. To be Poisson, these events must have the following properties:

- Events must occur one at a time. There are no simultaneous events. The number of events in a given period of times is **independent** of the number of events in any other period of time (that does not overlap the first). This property is often referred to as the *memoryless* or *Markov* property.

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[1]After Simeon-Denis Poisson (1781-1840) known as the inventor of the whoopee cushion.

Rates

Events in a Poisson process occur at some constant rate. If the rate is denoted by λ, then λ is expressed as a number of occurrences in a time period. For example, we might say that a given Poisson process has a rate of $\lambda = 5$ events per hour. The rate of a Poisson process has one very important property:

> Suppose that a Poisson process has a rate of λ. Then for a time period of length t, the probability of **exactly** one event in that period is approximated by λt. In particular, this estimate becomes very accurate for small t. If for example the rate of events is $\lambda = 5$ per hour, our formula says that the probability of exactly one event occurring in one hour is $\lambda t = 5 \cdot 1 = 5$, which makes absolutely no sense. If however, the time period is 1 minute (1/60'th of an hour) the probability of exactly one event occurring is $\lambda t = 5 \cdot (1/60) = 1/12$. In fact, 1/12'th is a good estimate of exactly one event occurring in one minute. Suppose now, that we are interested in the probability of exactly one event in one second. Then, $\lambda t = 5 \cdot (1/360) = 1/72$. Given a Poisson process with a rate of 5 events per hour, the probability of exactly one event in a second is indeed extremely close to $1/72 = .01388888....$. The exact answer is .013697... . The probability of two or more events in that one second is of course insignificant.

It follows, that to simulate Poisson events on a computer, one can divide the time into very small intervals. If customers arrive at a rate of 10 per hour we might divide our time period into one second intervals. In a single second, the probability of exactly one arrival is $10 \cdot (1/3600) = .000277777...$ (a second is 1/3600'th of an hour). Every second we draw a random number RND such that $0 \langle \text{RND} \langle 1$ and such that all numbers between 0 and 1 are equally probable. If RND 〈 .000277777 we say that a customer has arrived. This is not a particularly good simulation strategy and it is rarely used, but it is not entirely bad either.

Consequences of Poisson Assumptions

The most important assumption of the Poisson process is the assumption of independence. Since a Poisson process has no *memory*, the number of arrivals in any one period is independent of the number of arrivals in any other period (that does not overlap the first). Suppose that phone calls come into our interchange at the some known rate of λ per hour. Suppose also that we are analyzing a two-hour period. Let us denote by $P_1(n)$ the probability of n phone calls in a one-hour period. Similarly $P_2(n)$ will denote the probability of n phone calls in two-hours. The probability of say 20 phone calls in the first hour and 26 phone calls in the second hour is $P_1(20)\cdot P_1(26)$. Similarly it is easy to see that[1]

$$P_2(5) = P_1(0)\cdot P_1(5) + P_1(1)\cdot P_1(4) + P_1(2)\cdot P_1(3) + P_1(3)\cdot P_1(2) + P_1(4)\cdot P_1(1) + P_1(5)\cdot P_1(0)$$

That is, 5 phone calls in two hours can happen in one of exactly six ways. For example, there might be no phone calls in the first hour and 5 in the second hour. There might be 1 phone call in the first hour and 4 in the second hour, and so on.

[1]For you majors in the mathematical sciences, this kind of expression is known as a *convolution*.

The Poisson Formula

It so happens that there is a simple formula that describes completely the sort of random process we have discussed.[1] If events occur according to a Poisson process with rate λ, then the probability of exactly n occurrences in a period of time t is:

$$P(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Formula 1 The Probability of Exactly n Occurrences in a Time Interval of Length t.

In the Poisson formula, e is the mathematical constant used as the base of natural logarithms. e is found on all scientific calculators and its value is roughly 2.7182818285... .

A particularly convenient way of computing Poisson probabilities is by using the recursive form.[2]

$$\begin{aligned} P(0) &= e^{-\lambda t} \\ P(n) &= \frac{\lambda t}{n} P(n-1) \end{aligned}$$

Formula 2 Recursive Form of the Above Formula

[1]When I say *simple* I realize the formula will not seem simple at all to those of you who are math-phobes. However, if after your initial panic you will just follow the text, you should be okay.

[2]The Poisson recursive formula is perfect for spreadsheets. In fact, good spreadsheets have a large supply of scientific functions *but not* the factorial function. The reason is that it is far more appropriate to use recursion than to use the factorial function directly. With a good spreadsheet one can build a table of Poisson values and will construct a graph of those values in the spreadsheet. One can then change the λ and t values (each in a single square) and watch the graph change instantly.

<u>Example</u> Suppose that congressman Slackheart makes true utterances according to a Poisson process with rate 1 true utterance per month. The probability of no true utterances in one month would be $\frac{e^{-1}1^0}{0!} = e^{-1} = .367879$. The probability of one true utterance in one month would be $\frac{e^{-1}1^1}{1!} = e^{-1}$. The probability of two true utterances in one month would be $\frac{e^{-1}1^2}{2!} = \frac{e^{-1}}{2} = .18394$ the table below shows the values for n (the number of true utterances in one month) from 0 to 10.

$$P(n, \lambda t) := \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}$$

$\lambda t := 1 \quad n := 0..10$

n	$P(n, \lambda t)$
0	0.3678794412
1	0.3678794412
2	0.1839397206
3	0.0613132402
4	0.01532831
5	0.003065662
6	0.0005109437
7	0.000072992
8	0.000009124
9	0.0000010138
10	0.0000001014

Figure 1 The Poisson Probabilities

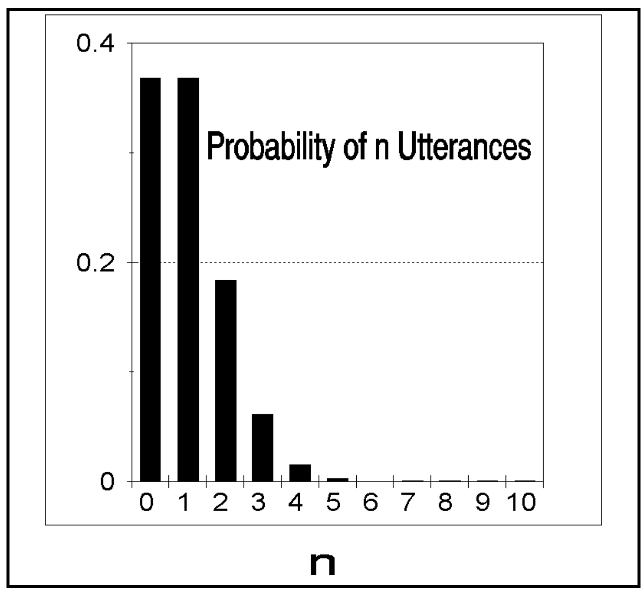

Figure 2 Graph of the Poisson Probabilities

Example Suppose that Professor Frankenstein gives coherent lectures according to a Poisson process with rate $\lambda = 2$ per month. What is the probability that he gives 3 coherent lectures in January given that he is known to give at least 2 coherent lectures that month? Remember, the Probability of A given B, written P(A|B) was defined to be P(A and B)/P(B). In this case, let us denote the event *n coherent lectures* (in one month) by C_n. We want $P(C_3|(C_2+C_3+C_4+...))$. The event $C_2+C_3+C_4+...$ is $1-C_0-C_1$. Hence $P(C_3|C_2)$ is $P(C_3)/P(1-C_0-C_1)$ which is:

$$\frac{P(C_3)}{P(1-C_0-C_1)} = \frac{\frac{e^{-5}5^3}{3!}}{1-\frac{e^{-5}5^0}{0!}-\frac{e^{-5}5^1}{1!}} = .1463$$

☐ **Exercise 1** Suppose that customers enter your restaurant at the rate of 10 per hour. What is the probability of exactly 10 arrivals in 1 hour; 9 arrivals; 11 arrivals; 0 arrivals; 20 arrivals?

☐ **Exercise 2** Consider the restaurant in the previous problem. What is the probability of 1 arrival in 10 minutes; of 20 arrivals in 2 hours; of 15 arrivals in 2 hours?

λ_1 and λ_2

In mathematics it seems that things conspire to be non-intuitive. However, one of the great virtues of Poisson processes is that they are intuitive in many ways. In particular, suppose we have a room with two doors and suppose that people come through door 1 according to a Poisson process with rate λ_1 and they come through door 2 according to a Poisson process with rate λ_2. Then, just as we would expect, the combined process is a Poisson process with rate $\lambda_1 + \lambda_2$.[1]

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[1]Whereas I think that proving this is a little much to require as an exercise for this book, (continued...)

The Exponential Distribution

The Poisson distribution[1] is considered a *discrete* distribution because it gives the probability of 0 events occurring or 1 event occurring or 2 events occurring but never a non-integer. The exponential distribution is called a *continuous* distribution. It gives the probability of an event occurring in non-integer units. However, the exponential and Poisson distributions are different aspects of the same thing.

Suppose that phone calls enter the Whiskey Exchange according to a Poisson process with an average rate of occurrence $\lambda t = 5$ per hour. Then the probability of exactly n phone calls in a period of t hours is: $P(n) = \frac{e^{-5t}(5t)^n}{n!}$. In particular the probability of 0 occurrences in a period of t hours is: $P(0) = e^{-5t}$. Hence the probability of **at least one** phone call in t hours is: $P(>0) = 1 - e^{-5t}$.[2] Now remember, a Poisson process is **memoryless**. The probability that a phone call occurs in (say) the next 10 minutes is independent of what has occurred in the past. In particular, since 10 minutes is 1/6'th of an hour, the probability that at

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[1](...continued)
it certainly is within reach of some of you. Try to set up the probability of n arrivals as a convolution (see earlier footnote this chapter). Then you need a little reorganization and you need the binomial theorem for $(a + b)^n$.

[1]I have not defined what a probability *distribution* is. However, what is important here is that when I talk about the Poisson distribution, I am referring to a situation where the Poisson formula applies.

[2]We are calculating 1 - P(0) which is much easier than calculating P(1) + P(2) + P(3) + P(4) +... ad infinitum. Note that these are the same thing.

least one phone call occurs in the next ten minutes is $P(>0) = 1-e^{-\frac{5}{6}}$. The probability that the next phone call occurs between t = a and t = b (we are at t = 0) is the probability that no phone call occurs between now and a, and that there is at least one phone call between t = a and t = b. In a Poisson process, non-overlapping periods are independent. Since the period of time from t = a to t = b is b − a we have that the probability that the next call occurs between t = a and t = b is: $e^{-\lambda a}(1-e^{-\lambda(b-a)}) = e^{-\lambda a}-e^{-\lambda b}$.

To summarize the preceding, in a Poisson process the time between events is described by the exponential distribution. Similarly, if the time between events is exponentially distributed with rate λ, then the events follow a Poisson process with rate λ.

$$\mathrm{P}(a,b) = e^{-\lambda a}-e^{-\lambda b}$$
$$0 \le a \le b \le \infty$$

The Exponential Distribution:
The Probability that the Next Event Occurs Between Times a and b

Typically when describing the exponential distribution, rather than saying that the time between events follows an exponential distribution with rate λ, one gives the mean time between events which happens to be 1/λ. For example: if customers arrive at our store according to a Poisson process with rate λ = 6 per hour, we frequently say instead, that the time between arrivals is exponentially distributed with mean time equal to 1/λ; that is a mean time between arrivals of 1/6 hours or 10 minutes. This should be intuitive: if arrivals occur at 6 per hour, the average time between arrivals is 1/6'th of an hour.[1] Remember that for the process to be Poisson, arrivals only occur one at a time.

A clerk might serve customers such that the time of service is exponentially distributed with a rate of λ. The average time of service is then 1/λ. We frequently think of such a process

[1]This is another case of the Poisson process behaving in an intuitive manner, which is again a matter of luck and not something to count on happening.

as Poisson with rate λ. But it is not really, because the clerk only serves customers when there are customers to be served. For example, the time of service might average 5 minutes. That corresponds to a Poisson process with rate 12 per hour. But unless there are always customers to be served, the clerk will serve an average of fewer than 12 customers per hour. Therefore, it is customary to describe arrivals as following a Poisson process and the service as being exponentially distributed.

Example A bank clerk serves customers (when the line is not empty) according to a Poisson process with rate $\lambda = 15$ per hour. The time of service is thus distributed according to an exponential distribution with mean time of service equal to 4 minutes. The probability that a given service will take between a and b hours is $e^{-5a} - e^{-5b}$. Thus the probability that the next service will take 10 to 20 minutes is $e^{-5/6} - e^{-5/3} = .4346 - .1889 = .2457$. The probability that the next service will take at least 10 minutes is $e^{-5/6} - e^{-\infty} = e^{-5/6} = .4346$. The probability that the next service will take no more than 20 minutes is $e^{0} - e^{-5/3} = 1 - .1889 = .8111$. A very important fact is this. Since the process is memoryless then the probability that the present service will last no more then 20 minutes is also .8111. The amount of time used up to now is irrelevant! All of the other calculations also hold for the present service.

Example Professor Ripper grades tests according to an exponential process with mean time of grading equal to 14 days. The rate of grading is thus $\lambda = 1/14$ per day (on average he grades 1/14 of the tests per day). The probability that a particular test will take him more than 10 days to grade is $e^{-\lambda 10} - e^{-\infty} = e^{-10/14} - e^{-\infty} = e^{-10/14} = .4895$.

☐ **Exercise 3** Students complain about Dr. Frankenstein according to a Poisson process with rate 9 per month. What is the mean time between complaints?

What is the probability that the next complaint will occur in the next two days? What is the probability that the next complaint won't occur until after two weeks? What is the probability that the next complaint occurs in the next 6 hours? What is the probability that the next complaint will occur between 5 to 10 days from now?

An Observer's Paradox <u>**This is Really Interesting!**</u>

The memoryless characteristic of the exponential distribution (and the Poisson distribution) leads directly to a paradox that is quite general and very important. Suppose that we have a worker who claims that the time to completion of each task is exponential with mean time of completion 3 hours ($\lambda = \frac{1}{3}$ per hour). The boss comes at completely random times and sees how long the task being performed requires to finish. This aspect of the problem is key: were the boss to wait until the next task started and to measure its length, the paradox would not occur. However, the boss shows up, and the task has already been going on for some time x. However, since the process is memoryless, the expected time to completion is still 3 hours, just as if the task started when the boss arrived. Hence on average, the tasks the boss observes will require more than 3 hours.

Solution to the Paradox

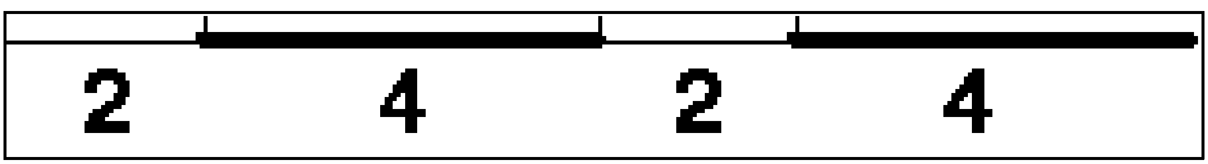

Figure 3 A Sequence of Two and Four Hour Jobs

The solution to paradox can be seen from an example using non-exponentially distributed times. This example accurately reflects the problem of *observer bias*. Consider the case where a worker performs tasks that alternately require 2 hours and 4 hours to perform, so that again the average time of performance is 3 hours. This situation is shown in **Figure 3**. In the box there are four tasks whose average is 3 hours. If we choose a time of arrival at random from the 12 hour period, the probability is $\frac{2}{3}$ that we will arrive during a 4 hour job, and the probability is $\frac{1}{3}$ that we will arrive during a 2 hour job. Then the expected length of the job is $\frac{2}{3}\cdot 4 + \frac{1}{3}\cdot 2 = 10/3 = 3.33....$ In this particular case (of alternating 2 and 4 hours jobs) the average length of a job is 3 hours, but the average job length to the outside observer who arrives at random times is 3.333 hours.

A similar situation occurs when estimating the average number of customers in a restaurant. To the customer, the average size will be larger than it is to the restaurant worker. The worker sees the restaurant at all times regardless of the number of customers. But customers are more likely to see the restaurant when it is busy. Therefore the worker and the customer will have contradictory perceptions of the amount of business.[1]

[1]Another example of a similar nature is the calculation of average classroom size. Suppose for example, that the school has two classes, one of size 100 and the other of size 2. From the school's point of view, the average classroom size is $(100 + 2)/2 = 51$. However, when we ask the students, 100 report a class size of 100 and 2 report a class size of 2 and this gives an average of $(100 \cdot 100 + 2 \cdot 2)/102 = 98.08$. Now the school can claim that their's is the correct average; the students have actually computed the harmonic mean rather than the arithmetic mean. The students can retort, justifiably I think, that the harmonic mean is the appropriate one to use.

1. For this problem $\lambda = 10$/hour and t = 1 and so $\lambda t = 10$. For n = 10 we get p = .12511; for n = 9 p = .12511; for n = 11 p = .11374; for n = 0 p = .0000454; for n = 20 p = .00187.

2. Again, $\lambda = 10$/hour. If t = 10 minutes, then t = 1/6 hours and $\lambda t = 10/6$; then for n = 1 p = .31479. When t = 2 hours, $\lambda t = 20$. For n = 20 p = .08884; for n = 15 p = .05165.

3. The rate of complaints is 9 per month or 9/30 = .3 per day. The average time between complaints is 1/9 months or 30/9 = 3.333 days. The probability that the next complaint occurs in the next 2 days is $1 - e^{-.3 \cdot 2} = 1 - e^{-.6} = .4512$. The probability that the next complaints doesn't occur for two weeks is $e^{-.3 \cdot 14} = e^{-4.2} = .0150$. The probability that the next complaint occurs in the next 6 hours is $1 - e^{-.3 \cdot .25} = .0722$. The probability that the next complaint occurs between 5 and 10 days from now is $e^{-.3 \cdot 5} - e^{-.3 \cdot 10} = .1733$.