33

The Gambler's Ruin[1]

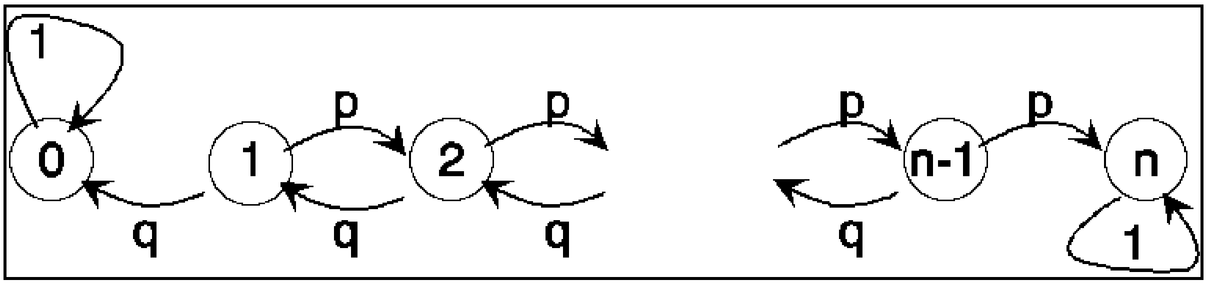

Figure 1 The Gambler's Ruin

There are many variations of the gambler's ruin problem, but a principal one goes like this. We have i dollars out of a total of n. We flip a coin which has probability p of landing heads, and probability $q = 1 - p$ of landing tails. If the coin lands heads we gain another dollar, otherwise we lose a dollar. The game continues until we have all n dollars or we are broke. The game is represented in **Figure 1**. For example suppose we have \$5 out of a total of \$15 but the coin is biased to produce heads .55 of the time. Is the bias of the coin sufficient to compensate for the fact that we have only ⅓ of the total funds? Even if the answer is clear to you, there are cases where your judgement is simply insufficient. That is why it is of interest to develop a formula for the gambler's ruin: Given, our funds, \$i, and the total, \$n, and the bias of the coin, p, what is the probability that we will win the whole amount?

In the graph there are $n + 1$ states corresponding to our possible fortune; that is state k is the case where we have k dollars. States 1 through $n - 1$ are transient states and states 0 and n are absorbing states since that is when the game ends. Markov chains consisting of just transient and absorbing states will be treated in Section 35. However, the gambler's ruin is interesting in its own right and will be solved separately here.

[1]Also known as the story of my life.

Let P_j denote the probability that starting in state j (that is starting with j dollars) we win all n dollars. Immediately we have $P_0 = 0$ and $P_n = 1$. If we start with \$1 we have probability q of going broke immediately and probability p of going to \$2 where our probability of winning it all is P_2. Hence, $P_1 = qP_2$. Similarly, if we have \$n−1, there is a probability of p of winning it all immediately, and there is a probability of q that we go down to \$n−2 where our probability of winning is P_{n-2}. Hence, $P_{n-1} = p + qP_{n-2}$. Similarly, if we have \$j where $1 < j < n-1$, there is a probability of p that our fortune improves to \$j+1 and there is probability of q that our fortune decreases to \$j−1. In other words:[1]

$$
\begin{aligned}
&P_0 = 0; \quad P_n = 1 \\
&P_1 = pP_2; \quad P_{n-1} = p + qP_{n-2} \\
&P_j = pP_{j+1} + qP_{j-1}; \quad 1 < j < n-1
\end{aligned}
$$

Formula 1 The Transition Probability Equations for Gambler's Ruin

Using the fact that $p + q = 1$, we can rewrite the last equation as $(p + q)P_j = pP_{j+1} + qP_{j-1}$, or $p(P_{j+1} - P_j) = q(P_j - P_{j-1})$. This we rewrite as $(P_{j+1} - P_j) = \rho(P_j - P_{j-1})$ where $\rho = q/p$.

[1]The derivation of the formulas in the box might be argued as heuristic and not rigorous. Whereas this is not intended as a particularly rigorous work, we can derive these formulas more carefully. I will do the third and most important formula; the others can be derived as special cases of this. If we are in state j, that is we have \$j, we want to find the probability of the event E_j, that we go on to win all of the money. To do this, we will define two exclusive and exhaustive events. First, there is the event, W, where we win the first bet. Second there is the event, L, where we lose the first bet. By the total law of probability $P(E_j) = P(E_j \cdot W) + P(E_j \cdot L)$. We can rewrite this as $P(E_j) = P(E_j|W)P(W) + P(E_j|L)P(L)$. In the text, the event $P(E_j)$ is denoted P_j; the event $P(E_j|W)$ is denoted P_{j+1}; the event $P(E_j|L)$ is denoted P_{j-1}; we have $P(W) = p$; and we have $P(L) = q$.

We then have:

- $P_2 - P_1 = \rho(P_1 - P_0) = \rho P_1$
- $P_3 - P_2 = \rho(P_2 - P_1) = \rho^2 P_1$
- $P_4 - P_3 = \rho(P_3 - P_2) = \rho^3 P_1$
- .
- .
- $P_n - P_{n-1} = \rho(P_{n-1} - P_{n-2}) = \rho^{n-1} P_1$

Solving the first equation in the list for P_2, we get $P_2 = P_1(1 + \rho)$. Solving the second equation for P_3, we get $P_3 = P_2 + \rho^2 P_1 = P_1(1 + \rho + \rho^2)$. In general, for $j \geq 1$, we get: $P_j = P_1(1 + \rho + \rho^2 + \rho^3 + ... + \rho^{j-1})$. Note that the expression within parentheses on the right is a finite geometric series. This gives us:

$$P_j = P_1(1 + \rho + \rho^2 + \rho^3 + ... + \rho^{j-1}) = P_1\left(\frac{1 - \rho^j}{1 - \rho}\right); \quad \rho = \frac{q}{p} \neq 1$$

Clearly the formula $\frac{1 - \rho^j}{1 - \rho}$ makes no sense if $\rho = 1$. However, if $\rho = 1$, then $1 + \rho + \rho^2 + \rho^3 + ... + \rho^{j-1} = j$. Remember, $\rho = 1$ only if $p = q = ½$. Now if $j = n$, we have $P_n = P_1\left(\frac{1 - \rho^n}{1 - \rho}\right)$. But since $P_n = 1$ we can solve for $P_1 = \frac{1 - \rho}{1 - \rho^n}$ with $P_1 = \frac{1}{n}$ in the case where $\rho = 1$. Substituting for P_1 in the formula for P_j, we get:

$$P_j = \left(\frac{1 - \rho^j}{1 - \rho^n}\right); \quad \rho = \frac{q}{p} \neq 1$$

$$P_j = \frac{j}{n}; \quad \rho = \frac{q}{p} = 1$$

Formula 2 The Solution to the Gambler's Ruin

One result is quite elegant. If you and I are gambling over a total of \$100 and I have \$13 of the total, and if we are betting \$1 at a time **fairly,** my probability of winning is .13. In general if we are gambling over a finite amount of resources, the person in the lead has an advantage (which might be negated if $p \neq q$).

☐ **Exercise 1** Solve the problem given in the first paragraph: We have \$5 out of a total of \$15. We are flipping a coin, and when we get heads, we win a dollar. The coin happens to be biased to produce heads .55 of the time.[1] What is our probability of winning all \$15?

☐ **Exercise 2** A drunk is ambling along a narrow walkway. In general, he makes two steps forward for every three backwards. Going forwards he is only 10 steps from his office (in the university administration building). In the other direction, he is 50 feet from the campus police office (and a hefty donation to the Police officer's Benevolence Fund). What is his probability of making it to his office?

☐ **Exercise 3** Use our analysis of gambler's ruin to answer another type of gambler's ruin problem. You have \$100, and your only chance of raising the \$500 you need for your mother's operation is to gamble for it. Each bet pays one-to-one, but your probability of winning each bet is only .48. Should you bet large amounts, or should you bet the minimum bet of 1\$. Having answered that, what is the probability of raising the necessary money? (This is also an excellent type of problem to simulate on a computer. Using a high-level language such as Pascal, and a random number generator, you should be able to encode this simulation in about 30 lines of code or less.)

[1]We had no idea that the coin was biased. We, in fact, are flabbergasted. We have no idea how we came to be in the possession of a biased coin.

1. $\rho = .8181...$; $n = 15$; $P_5 = .666188$

2. $\rho = 1.5$; $n = 60$; $P_{50} = .017342$

3. Playing with the formula for gamblers ruin, you should see that when the odds are in your favor, you want to make small bets. When the odds are against you, you want to make large bets.

   As for the probability of getting the money you need, you can solve this problem analytically by modeling it as a Markov chain. The methodology will be developed in the next chapters. The other way to solve it is by simulation. That goes as follows: (Remember: whether solving this problem by analysis or simulation, since the odds are against you, at each stage you bet all the money you have up to what you need!) The strategy used in either analysis is that you bet the minimum of what you need versus what you have. For example, if you have \$60 and you need \$50,you bet \$50. If you have \$50 and you need \$60 you bet \$50 (this case is not so clear, however you <u>do not</u> bet \$1 at a time).

   Simulation for Estimating the probability of winning \$400 with stake of \$100.

```
p ← .48                        {probability of winning each bet}
number_of_wins ← 0  {counter}
                    {rand is a random variate uniform between 0 and 1}

FOR i ← 1 TO 1000   {we will do 1000 simulations}
        BEGIN
                funds ← 100                 {starting funds}
                WHILE funds 〉 0 AND funds 〈 500
                        BEGIN
                                bet ← min(500-funds, funds)
                                IF rand ≤ .48 THEN funds ← funds + bet
                                        ELSE funds ← funds − bet
                        END {while}
          IF funds = 500 THEN number_of_wins ← number_of_wins + 1
        END
output number_of_wins/1000
```