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The Bonferroni Inequality

The Bonferroni inequality is a fairly obscure rule of probability that can be quite useful.[1]

$$\boxed{P(a_1 a_2 \ldots a_n) \geq P(a_1) + P(a_2) + \ldots + P(a_n) - n + 1}$$

The Bonferroni Inequality

The proof is by induction. The first case is n = 1 and is just $P(a_1) \geq P(a_1)$. To just be sure, we try n = 2: $P(a_1 a_2) \geq P(a_1) + P(a_2) - 1$. To prove this we note that $1 \geq P(a_1 + a_2)$. However, the law of addition says: $P(a_1 + a_2) = P(a_1) + P(a_2) - P(a_1 a_2)$. Substituting this last identity in the previous one, we move the 1 to the right of the inequality and we move the $P(a_1 a_2)$ to the left of the inequality and this gives us the desired result. Lastly, we have to do the inductive step. We assume that the proposition is true for n (as in the box) and we then show that it necessarily follows for the case n + 1. Now we refer to the case n = 2 that we have just proven. We get: $P(a_1 a_2 a_3 \ldots a_n a_{n+1}) \geq P(a_1 a_2 a_3 \ldots a_n) + P(a_{n+1}) - 1$. Now if we substitute for $P(a_1 a_2 a_3 \ldots a_n)$ using the induction hypothesis (the box) we get:
$P(a_1 a_2 \ldots a_{n+1}) \geq P(a_1) + P(a_2) + \ldots + P(a_{n+1}) - n$ which is what had to be shown.

Example Suppose that we have ten events a_i, with $P(a_i) = .99$. We want to estimate the joint probability $P(a_1 a_2 \ldots a_{10})$. If the a_i are independent events then we have: $P(a_1 a_2 \ldots a_{10}) = P(a_1)P(a_2)\ldots P(a_{10}) = .99^{10} = .904382075009$. However we have no grounds to assume independence. If we use the Bonferroni inequality get:

$$\begin{aligned} &P(a_1 a_2 \ldots a_n) \geq P(a_1) + P(a_2) + \ldots + P(a_n) - n + 1 \\ &= 10(.99) - 9 = .9 \end{aligned}$$

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[1] By far its most useful application is in joint confidence intervals. The inequality gives you a confidence interval without assuming independence of the various parameters. It usually turns out at around 95% confidence that the confidence region isn't much smaller than with the assumption of independence.

Note how close the two numbers are.

☐ **Exercise 1** Repeat the above example with $P(a_i) = .1$ for each I.

1. If the events were independent the joint probability would be $(.1)^{10} = 10^{-10}$. Bonferroni's inequality says that the joint probability is greater or equal to $10(.1)-9 = -8$. This of course is useless. The Bonferroni inequality is useful as the probabilities of the events get larger.