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The Sum of Three Die:

An Application of Binomial Coefficients

(Optional)

To solve probability problems involving the sum of two die, we constructed a table containing all 36 outcomes. To solve problems involving the sum of three die, we could construct a three-dimensional table with all 216 outcomes ($6^3 = 216$). However, it would be nice if we could find a quicker method.

The sum of three die must range from 3 (1 plus 1 plus 1) to 18 (6 plus 6 plus 6). Since each of these two sums can occur in only one way, we have that the probability of each is $\frac{1}{216}$.

We can also see that just as in the two-die case, there is symmetry: there are as many ways to get 17 as to get 4; to get 16 as to get 5. Finally, in the center the most likely sums are for 10 and 11.[1] Therefore we need only calculate the number of ways to achieve 3 though 10. For 11 through 18 we get the same calculations but in the reverse order.

Let us calculate the number of ways three die can sum to 8. Three such combinations of three die might be 4-2-2, 2-4-2 and 3-3-2. Let us denote each of these three combinations as a partition of the 8 squares, □□□□□□□□, into three classes. 4-2-2 is represented by □□□□|□□|□□. 2-4-2 is represented by □□|□□□□|□□. 3-3-2 is represented by

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[1]If you want a formal argument as to the symmetry of the sum of three die use the following method: subtract from every face of each die 3.5. The die now range from a low of −2.5 to a high of 2.5. Each sum of these die corresponds in the original faces to the same sum plus 10.5 (which is three times 3.5). Now for every combination with a positive sum there corresponds a combination with a negative sum. This demonstrates the symmetry of the distribution.

□□□|□□□|□□. Each combination that adds to eight is represented by marking two of the seven spaces between the 8 squares. The number of combinations of three die that add to 8 is

$\binom{7}{2} = 21$. Hence the probability of rolling an 8 is $\frac{21}{216} = \frac{7}{72}$. By symmetry the

probability of rolling a 13 is also $\frac{7}{72}$.

We can use the above method to calculate the number of combinations of the die that add to 3 through 8 (and by symmetry 13 through 18). We will denote the number of ways of summing to c by N(c). For example, N(8) = 21. The probability of rolling a c (where c is an integer between 3 and 18 inclusive) is $\frac{N(c)}{216}$. These numbers are:

▸ $\quad N(3) = \binom{2}{2} = 1 = N(18)$

▸ $\quad N(4) = \binom{3}{2} = 3 = N(17)$

▸ $\quad N(5) = \binom{4}{2} = 6 = N(16)$

▸ $\quad N(6) = \binom{5}{2} = 10 = N(15)$

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▸ $\quad N(7) = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15 = N(14)$

▸ $\quad N(8) = \begin{pmatrix} 7 \\ 2 \end{pmatrix} = 21 = N(13)$

We have calculated the number of ways three die can add up 3 through 8 and 13 through 18.   But what about 9, 10, 11, and 12?   If we apply our method to find N(9) we get

$N(9) = \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 28$. However N(9) does not equal 28. Three of the combinations we counted

are: □|□|□□□□□□□ which corresponds to the combination $1-1-7$; □|□□□□□□□|□ which corresponds to the combination $1-7-1$; □□□□□□□|□|□ which corresponds to the combination $7-1-1$. These combinations are not valid because each involves a die showing a 7. It is easy to see that these are the only combinations adding to 9 using a 7. Similarly no combination can add to 9 using an 8, because the smallest sum involving an 8 adds to 10 (such as $8-1-1$). Hence $N(9) = 28-3 = 25$ and by symmetry $N(12) = 25$. This leaves calculating N(10) and N(11) which

by symmetry must be equal. Our method is to use $\begin{pmatrix} 9 \\ 2 \end{pmatrix} = 36$. The sums we must eliminate

are those involving die with faces equal to 7 or 8. Three die with one having a face greater than 8 cannot sum to 10. It is easy to see that the invalid combinations are of two types. First there are sums using a 7, a 2 and a 1. Second there are sums involving an 8 and two 1's. There are

$3! = 6$ ways of permuting 7, 2, and 1. There are $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$ arrangements of an 8 and two 1's

(these are the number of ways you can place the 8). The other two slots get the 2's. Altogether

we have $6 + 3 = 9$ sums that are not valid. Hence $N(10) = N(11) = \begin{pmatrix} 9 \\ 2 \end{pmatrix} - 9 = 27$. As a check

of our calculations, you can sum N(3) through N(18) to get 216.

The probabilities for the sums of three die are:

c	N(c)	P(c)
3	1	0.0046
4	3	0.0139
5	6	0.0278
6	10	0.0463
7	15	0.0694
8	21	0.0972
9	25	0.1157
10	27	0.1250
11	27	0.1250
12	25	0.1157
13	21	0.0972
14	15	0.0694
15	10	0.0463
16	6	0.0278
17	3	0.0139
18	1	0.0046
SUM	**216**	**1**