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## Probability and the Law of Addition

### Notation

We are interested in the probabilities of *events*. Whatever events are, they are not numbers. Therefore, given events  $A$  and  $B$ , if I speak of the event  $A + B$ , it does not mean  $A$  plus  $B$ . Rather it is shorthand for  $A$  or  $B$ . Elsewhere, but not in this book, it is denoted by  $A \cup B$  or  $A \vee B$ . Similarly, we will denote  $A$  and  $B$  by  $A \cdot B$  or just  $AB$ . Elsewhere it is denoted by  $A \cap B$  or  $A \wedge B$ . We will denote *not*  $A$  by  $\bar{A}$ . Elsewhere, it is denoted by  $A'$  or  $\sim A$ .  $A - B$  means  $A$  but not  $B$  which is the same as  $\bar{A}B$ :  $A$  and not  $B$ . Notice that this is the terminology and notation of sets, as in section 19. The reason goes as follows: Given a probabilistic experiment we construct a *sample space* of all the atomic outcomes. By *atomic* outcomes, I mean indivisible outcomes. For example, I want to toss a coin four times and record whether it comes up heads and tails on each throw. Then the sample space consists of all  $2^4 = 16$  possible outcomes. One such outcome would be  $httt$ : heads followed by three tails. Another would be  $htht$ : heads followed by tails, then heads then tails. Now we could ask the probability associated with each of these outcomes. Note that to compute the answers, we would need more information and this will be covered later. However, we might be interested in an event that is a subset of the sample space. We might be interested in the event that we throw more heads than tails. This event can be written  $httt + thtt + ttht + ttth + tttt$  (again the plus sign in this context means *or*). So now we see that a probabilistic event may be any subset of the sample space for the experiment, and this works as a definition of *event*.

We denote the probability of an event,  $X$ , by  $P(X)$  whereas elsewhere it is sometimes denoted  $\Pr(x)$ .  $AB$ , or  $A$  and  $B$ , is a *joint* event, and its probability,  $P(AB)$ , is a *joint probability*. All we need to know about probabilities now, is that the probability of some event is a real

number between 0 and 1 inclusive. A probability of 0, means that something is not possible. A probability of 1 means that an event is certain.<sup>1</sup> Since  $\bar{X}$  is *not*  $X$  we have that  $P(\bar{X}) = 1 - P(X)$ . These rules and the law of addition which follows are the basis of our work. The law of multiplication that we see in Section 23 will be based upon a definition—the definition of conditional probability.

**Remember, if you see  $P(A) + P(B)$ ,  $P(A)$  and  $P(B)$  are numbers and they can be added. If you see  $P(A + B)$ ,  $A$  and  $B$  are events, and  $A + B$  is just shorthand for  $A$  or  $B$ .**

## The Laws of Events

Events obey certain laws which I will state here **for reference**. These laws are essentially the same laws as just given in Section 19. I do not expect you to use many of these, but occasionally they can be handy. The laws are the laws of Boolean algebra. Alternatively (and equivalently) they can be viewed as the laws of sets. This is because as we have just seen events are subsets of the sample space. In this text I have usually avoided reference to sets and instead have used the less formal term *collections*. The reason is that we don't really get much mileage out of set conventions (in this text). The connection between events and sets are that they behave the same; that is they obey the laws of a Boolean algebra. The reason is that events can be looked on as subsets of sample spaces. Again, these laws are here **for reference**. Following some laws I will provide an informal paraphrasing to try to make the law seem intuitive. Lastly, we need to introduce two special events: the universal event,  $U$ , and the empty event,  $\emptyset$ . These have the properties that  $P(U) = 1$  and  $P(\emptyset) = 0$ . However, the following laws are about events and say nothing directly about probabilities. The event  $U$  can sometimes be

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<sup>1</sup>Don't write to me about sets of measure zero, or nonstandard analysis and infinitesimal probabilities. This is strictly an elementary text.

thought of as *truth*, or as the event *something happens*. Note that in this context  $U$  is actually the sample space. The event  $\emptyset$  can be thought of as *falsity*, or as an impossible event (such as the Detroit Lions win the Superbowl). The laws of events are:

- ▶ Associativity:  $A(BC) = (AB)C$ ;  $A + (B + C) = (A + B) + C$
- ▶ Commutivity:  $AB = BA$ ;  $A + B = B + A$
- ▶ Distribution:<sup>1</sup>  $A(B + C) = AB + AC$ ;  $A + (BC) = (A + B)(A + C)$
- ▶ DeMorgan's Laws:  $\overline{(A+B)} = \overline{A}\cdot\overline{B}$ ;  $\overline{(A\cdot B)} = \overline{A} + \overline{B}$  : The DeMorgan laws are not

obvious but can be arrived at by reasoning. The first law says that neither  $A$  nor  $B$  is the same as not  $A$  and not  $B$ . The second law says that not ( $A$  and  $B$ ) is the same as not  $A$  or not  $B$ .

- ▶ Negation:  $\overline{\overline{A}} = A$  : Not not  $A$  is just  $A$ .
- ▶ Identity:  $A + \emptyset = A$ ;  $A \cdot U = A$ : “ $A$  will happen *or* an impossibility will happen” means that  $A$  will happen. “ $A$  will happen *and* something will happen means” that  $A$  will happen.
- ▶ Idempotency:  $A + A = A$ ;  $A \cdot A = A$ : “ $A$  **or**  $A$  will happen” means that  $A$  will happen. “ $A$  **and**  $A$ ” will happen means that  $A$  will happen.
- ▶ Absorption:  $A \cdot \emptyset = \emptyset$ ;  $A + U = U$ : “ $A$  will happen **and** an impossibility will happen” is impossible. “ $A$  will happen **or** something will happen” is the same as something will happen.
- ▶ Complementarity:  $A \cdot \overline{A} = \emptyset$ ;  $A + \overline{A} = U$ : “ $A$  **and** not  $A$ ” is impossible. “ $A$  **or** not  $A$ ” is certain.

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<sup>1</sup>Boolean algebras are virtually the only algebras where addition distributes over multiplication.

## Counting and Probability

In Section 15 we solved some probability problems involving poker hands simply by counting. The principal behind this can be best illustrated by considering the fair toss of a fair pair of die:

**The Outcomes of the Toss of Two Die**

1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

There are 36 outcomes to the throw of two standard die, and these are in the fact the sample space. By assuming that the die and the throw are fair, we infer that each outcome is equally likely. In the table above each cell represents a single outcome; the first entry in a cell is the outcome of a distinguished first die, and the second entry is the outcome of the other die. The probability of any particular outcome is  $1/36$ .

Let us denote by  $F_i$  the event that the first die has value  $i$ , where  $i$  can be 1 through 6. Let us denote by  $S_i$  the event that the sum of the die is  $i$ , where  $i$  can be 2 though 12. The probability,  $F_4$ , that the first die should be a 4 is given by the events of the fourth column, that is 6 out of 36 outcomes. Hence,  $P(F_4)$  is  $6/36$  or  $1/6$ , just as it should be. The event,  $S_7$ , that the sum of the die is a 7 is the sequence of outcomes in one of the main diagonals; that is 6 cases out of 36. Hence  $P(S_7) = 1/6$ .

To solve for the joint probability  $P(S_3F_2)$ , we observe that only one outcome has sum 3 with the first die 2. Hence,  $P(S_3F_2) = 1/36$ . Similarly,  $P(S_2F_3) = 0$ . Without even looking we know that  $P(F_2F_3) = 0$  and that  $P(S_2S_3) = 0$ . Therefore, we know one statement that is **not** always

true:  $P(XY) = P(X)P(Y)$ . The general treatment of joint probabilities requires some more definitions and will be handled in the Section 23. However, we can treat the general case of  $P(X + Y)$  now.

## The Law of Addition

The *law of addition* refers to  $P(X + Y)$ , the probability of  $X$  or  $Y$ . Again, events cannot be added. We can demonstrate the law by a particular case. Let us consider the probability that the first die is a 3 **or** the sum of the die is 6. That is, what is  $P(F_3 + S_6)$ ? Again, let us look at all the possible outcomes:

The Event First Die is 3 **or** the Sum is 6 is in Grey

1, 1	2, 1	3, 1	4, 1	5, 1	6, 1
1, 2	2, 2	3, 2	4, 2	5, 2	6, 2
1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
1, 4	2, 4	3, 4	4, 4	5, 4	6, 4
1, 5	2, 5	3, 5	4, 5	5, 5	6, 5
1, 6	2, 6	3, 6	4, 6	5, 6	6, 6

Simply by counting cells, we can calculate that  $P(F_3 + S_6) = 10/36 = 5/18$ . We would like that  $P(F_3 + S_6) = P(F_3) + P(S_6)$  simply because it looks like a keen law that would make life a lot simpler. However, in this case  $P(F_3) = 1/6$  and  $P(S_6) = 5/36$ , and  $1/6 + 5/36 = 11/36 \neq 10/36$ . The reason that we don't have  $P(F_3 + S_6) = P(F_3) + P(S_6)$  is pretty easy to see. The two events  $F_3$  and  $S_6$  each have an outcome in common, which is  $F_3S_6$ .  $F_3S_6$  is the event that the first die is 3 **and** the sum is 6. This is simply the case where each die is 3. If we add the probabilities of  $F_3$  and  $S_6$ , we add the probability of this case twice.

The above example is strong grounds for a law which we take as an axiom:

$$\mathbf{P(X + Y) = P(X) + P(Y) - P(XY)}$$

### The Law of Addition

This law is *always* true.  $P(X + Y) = P(X) + P(Y)$  only if  $P(XY) = 0$ ; in that case (that  $P(XY) = 0$ ) we say that the events  $X$  and  $Y$  are *disjoint*. The law of addition can be generalized to an arbitrary number of events. In the case of three events, we have  $P(A + B + C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$ .<sup>1</sup> Rather than learning a generalized law, you can do multiple events recursively. For example,  $P(A + B + C) = P((A + B) + C) = P(A + B) + P(C) - P((A + B)C) = P(A + B) + P(C) - P(AC) - P(BC)$ . After we use the law of addition (again) to evaluate  $P(A + B)$ , we have recreated the three-event formula I just gave.

- **Exercise 1**      The probability that Sam and Sherry go to a movie tonight is .4. The probability that it rains tonight is .3. The probability that Sam and Sherry go to a movie tonight and it rains is .1. What is the probability that Sam and Sherry go to a movie tonight or it rains?
- **Exercise 2**      Suppose that  $P(A) = .7$  and  $P(B) = .6$ , what can we say about  $P(AB)$ ?  
Hint: We do not have enough information to say exactly what  $P(AB)$  is.

**Example**      In the game Parcheesi, to take someone else's game piece you can use either die or the sum of both die. This means that the probabilities of various numbers are different than how they are usually calculated. Suppose you need to throw a 3. You can get a 3 with either die or with the sum of the die. The probability of

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<sup>1</sup>The general case of the law of addition is derived from the *principal of inclusion and exclusion* which is the next section in this book.

getting it with either die is  $\frac{1}{3} + \frac{1}{3} - \frac{1}{36} = \frac{11}{36}$ . The probability of getting 3 as a sum

is  $\frac{1}{18}$ . These are disjoint events, so the probability of one event or the other is

their sum:  $\frac{13}{36}$ .

- Exercise 3**      Compute the probabilities for all of the possible throws in Parcheesi.
- Exercise 4**      Why do the probabilities in the previous exercise add up to more than 1?

1. We have:  $P(\text{movie}) = .4$ ;  $P(\text{rain}) = .3$ ;  $P(\text{movie and rain}) = .1$ . We want  $P(\text{movie or rain}) = P(\text{movie} + \text{rain})$ . By the law of addition,  $P(\text{movie} + \text{rain}) = P(\text{movie}) + P(\text{rain}) - P(\text{movie and rain}) = .4 + .3 - .1 = .6$ .
2.  $AB$  is the event  $A$  and  $B$  and like any other event we must have  $0 \leq P(AB) \leq 1$ . Also we must have  $P(AB) \leq P(A), P(B)$ . Hence we know to start with that  $0 \leq P(AB) \leq .6$ . However, we know also that  $0 \leq P(A + B) \leq 1$ . In other words,  $0 \leq P(A) + P(B) - P(AB) \leq 1 \Rightarrow 0 \leq .7 + .6 - P(AB) \leq 1 \Rightarrow 0 \leq 1.3 - P(AB) \leq 1 \Rightarrow -1.3 \leq -P(AB) \leq -.3 \Rightarrow .3 \leq P(AB) \leq 1.3$ . Combining this with what we already have, we get  $.3 \leq P(AB) \leq .6$ .

3.

Throw	Probability
1	11/36
2	1/3
3	13/36
4	7/18
5	5/12
6	4/9
7	1/6
8	5/36
9	1/9
10	1/12
11	1/18
12	1/36

4. All throws are counted more than once. For example, a 3-4 counts three times, as a 3, a 4, and a 7. A 3-3 counts twice, as a 3 and a 6.