Chapter 17 out of 37 from *Discrete Mathematics for Neophytes: Number Theory, Probability, Algorithms, and Other Stuff* by J. M. Cargal

17

Stirling Numbers of the Second Kind:
Counting Partitions

Again: a partition of n objects is a division of these objects into separate classes. Each object must be in one and only one class and partitions with empty classes are not allowed. A question we might easily ask is *how many ways can we partition n objects into k classes?* For example, *how many ways can we partition 4 objects into 2 classes?* This number is denoted \[ \left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\} \] and is called a *Stirling number of the second kind.* There is no agreed upon standard of notation for these number; other books display them differently.\(^1\) Note the similarity between Stirling numbers \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) and binomial coefficients \( \binom{n}{k} \). To solve for \( \left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\} \) we can simply enumerate all of the cases. This is done in Figure 1 which shows that \( \left\{ \begin{array}{c} 4 \\ 2 \end{array} \right\} = 7 \).

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Figure 1  The Partitions of 4 objects into 2 classes

Given that \( \{7\} = 350 \), enumeration of Stirling numbers is almost always impractical.

What is perhaps the easiest approach is recursion. Suppose we want to calculate \( \left\{ \begin{array}{c} m \\ k \end{array} \right\} \). That is, how many ways can we put \( n \) objects into \( k \) classes? Let us select a single object which we will call \( Fred \). We will now consider the two cases: either \( Fred \) occupies a partition class by himself or he is in a partition class with other objects. If he is in a class by himself, then there are \( \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} \) ways to put the other \( n-1 \) objects into the other \( k-1 \) classes. If \( Fred \) is not alone, there are \( \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} \) ways to put the other \( n-1 \) objects into \( k \) classes. Having done that, there are \( k \) choices (the \( k \) classes) where to place \( Fred \). This gives us the following recursive relation:
\[ \binom{n}{k} = \binom{n-1}{k-1} + k \binom{n-1}{k} \]. As always we need initial conditions to halt the recursion. This gives us the full recursion:

\[
\begin{align*}
\binom{n}{0} &= 1 \\
\binom{n}{1} &= 1 \\
\binom{n}{k} &= \binom{n-1}{k-1} + k \binom{n-1}{k} 
\end{align*}
\]

The Recursion for Stirling Numbers of the Second Kind

If we apply the above definition to compute \( \binom{5}{2} \) we get:

\[
\begin{align*}
\binom{5}{2} &= \binom{4}{1} + 2 \binom{4}{2} = 1 + 2 \left( \binom{3}{1} + \binom{3}{2} \right) = \\
1 + 2 \left( 1 + 2 \left( \binom{2}{1} + 2 \binom{2}{2} \right) \right) &= 1 + 2(1 + 2(1 + 2)) = 15
\end{align*}
\]
A table of the Stirling numbers of the second kind through \( \binom{10}{0} \) is given below.\(^1\)

<table>
<thead>
<tr>
<th>( k \backslash n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
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<tr>
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<td>1</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>31</td>
<td>63</td>
<td>127</td>
<td>255</td>
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<td>1</td>
<td>6</td>
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<td>301</td>
<td>966</td>
<td>3025</td>
<td>9330</td>
<td></td>
<td></td>
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<tr>
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<td>1</td>
<td>10</td>
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<td>350</td>
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<td>7770</td>
<td>34105</td>
<td></td>
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<td>140</td>
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</tbody>
</table>

\( \square \) Exercise 1  
Use the above table and the recursive formula to evaluate \( \binom{12}{11} \).

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\(^1\) An identical table is found in the book by Pólya, Tarjan, and Woods which I have referenced as an excellent book several times in this text. However, I generated my table using a spreadsheet. In fact spreadsheets are ideal for this purpose and it can be done quite swiftly.
1. \[
\binom{12}{11} = \binom{11}{10} + 11\binom{11}{11}
\]

but \[
\binom{11}{10} = \binom{10}{9} + 10\binom{10}{10} = 45 \text{ [from the table]} + 10.
\]
The final answer is then \[
45 + 10 + 11 = 66.
\]