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Stirling Numbers of the Second Kind: Counting Partitions

Again: a partition of n objects is a division of these objects into separate classes. Each object must be in one and only one class and partitions with empty classes are not allowed. A question we might easily ask is *how many ways can we partition n objects into k classes?* For example, *how many ways can we partition 4 objects into 2 classes?* This number is denoted $\left\{ {4 \atop 2} \right\}$ and is called a *Stirling number of the second kind.* There is no agreed upon standard of notation for these number; other books display them differently.[1] Note the similarity between Stirling numbers $\left\{ {n \atop k} \right\}$ and binomial coefficients $\binom{n}{k}$. To solve for $\left\{ {4 \atop 2} \right\}$ we can simply enumerate all of the cases. This is done in **Figure 1** which shows that $\left\{ {4 \atop 2} \right\}$ = 7.

[1] I am using D. E. Knuth's notation from his great work *The Art of Computer Programming*, Vol I, 3rd. ed. Addison-Wesley, 1997.

1	a b	c d
2	a c	b d
3	a d	b c
4	a	b c d
5	b	a c d
6	c	a b d
7	d	a b c

Figure 1 The Partitions of 4 objects into 2 classes

Given that $\left\{ {7 \atop 4} \right\}$ = 350, enumeration of Stirling numbers is almost always impractical.

What is perhaps the easiest approach is recursion. Suppose we want to calculate $\left\{ {n \atop k} \right\}$. That is, how many ways can we put n objects into k classes? Let us select a single object which we will call *Fred*. We will now consider the two cases: either Fred occupies a partition class by himself or he is in a partition class with other objects. If he is in a class by himself, then there are $\left\{ {n-1 \atop k-1} \right\}$ ways to put the other n−1 objects into the other k−1 classes. If Fred is not alone, there are $\left\{ {n-1 \atop k} \right\}$ ways to put the other n−1 objects into k classes. Having done that, there are k choices (the k classes) where to place Fred. This gives us the following recursive relation:

$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}$. As always we need initial conditions to halt the recursion. This gives us the full recursion:

$$\left\{ n \atop n \right\} = 1$$
$$\left\{ n \atop 1 \right\} = 1$$
$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}$$

The Recursion for Stirling Numbers of the Second Kind

If we apply the above definition to compute $\left\{ 5 \atop 2 \right\}$ we get:

$$\left\{ 5 \atop 2 \right\} = \left\{ 4 \atop 1 \right\} + 2\left\{ 4 \atop 2 \right\} = 1+2\left(\left\{ 3 \atop 1 \right\} + \left\{ 3 \atop 2 \right\} \right) =$$
$$1+2\left(1+2\left(\left\{ 2 \atop 1 \right\} + 2\left\{ 2 \atop 2 \right\} \right) \right) = 1+2(1+2(1+2\cdot 1)) = 15$$

A table of the Stirling numbers of the second kind through $\left\{ \begin{matrix} 10 \\ 10 \end{matrix} \right\}$ is given below.[1]

k\n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511
3			1	6	25	90	301	966	3025	9330
4				1	10	65	350	1701	7770	34105
5					1	15	140	1050	6951	42525
6						1	21	266	2646	22827
7							1	28	462	5880
8								1	36	750
9									1	45
10										1

☐ **Exercise 1** Use the above table and the recursive formula to evaluate $\left\{ \begin{matrix} 12 \\ 11 \end{matrix} \right\}$.

[1]An identical table is found in the book by Pólya, Tarjan, and Woods which I have referenced as an excellent book several times in this text. However, I generated my table using a spreadsheet. In fact spreadsheets are ideal for this purpose and it can be done quite swiftly.

1. $$\left\{ \begin{matrix} 12 \\ 11 \end{matrix} \right\} = \left\{ \begin{matrix} 11 \\ 10 \end{matrix} \right\} + 11 \left\{ \begin{matrix} 11 \\ 11 \end{matrix} \right\}$$

but $\left\{ \begin{matrix} 11 \\ 10 \end{matrix} \right\} = \left\{ \begin{matrix} 10 \\ 9 \end{matrix} \right\} + 10 \left\{ \begin{matrix} 10 \\ 10 \end{matrix} \right\}$ = 45 [from the table] + 10. The final answer is then 45 + 10 + 11 = 66.