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Multinomials

Let us slightly generalize the problem of selecting r out of n objects without respect to order. Suppose, for example, that I have 20 students. Motivated by the same sort of selfish consideration that I had in Chapter 5, I decide to divide the class evenly into A's, B's, C's, D's, and F's. The question we will answer here is: How many ways can I do this? We can answer this problem using binomial coefficients. There are $\binom{20}{4}$ of picking the students who get A's. This leaves $\binom{16}{4}$ ways of picking who gets B's. This in turn leaves $\binom{12}{4}$ ways to choose the C's, $\binom{8}{4}$ ways to choose the D's and $\binom{4}{4}$ ways of picking the F's. The product of these numbers is $4845\times1820\times495\times70\times1 = 305540235000$.

It turns out that it is useful to rewrite this product in terms of the formula $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. After cancellation we get $\frac{20!}{4!4!4!4!4!}$. In general by applying that same formula in this way we get:

> The number of ways of choosing $m_1, m_2, m_3, \ldots, m_k$ out of n objects is
>
> $$\binom{n}{m_1\ m_2\ m_3\ \ldots\ m_k} = \frac{n!}{m_1!\ m_2!\ m_3!\ \ldots\ m_k!}$$
>
> This is the *multinomial coefficient* n over $m_1, m_2, m_3, \ldots, m_k$.
> Where $m_1 + m_2 + m_3 + \ldots + m_k = n$.

Note that binomial coefficients are special cases of multinomial coefficients (corresponding to k = 2). However the notation is not quite consistent. In the above notation the binomial coefficient $\binom{6}{2}$ is denoted $\binom{6}{4\ 2}$. However, it should be clear that the meaning is the same.

☐ **Exercise 1** We have four people playing poker. Each person is dealt five cards. How many hands can there be dealt? (Is this problem precise?)

1. The problem is not stated precisely enough. It is tempting to suggest that the answer is:

$$\binom{52}{5\quad 5\quad 5\quad 5\quad 32} \approx 1.47826 \cdot 10^{24}$$

This is the number of ways that 52 cards can be dealt to four hands of five cards each. The key point is that each hand depends only on the five cards and not the order that they are dealt in. However, the game is determined also, by the arrangement of the hands. The play and the betting is done sequentially, and the order in which the hands occur is important. Hence the best answer would be to multiply the above answer by 4! (which is the number of permutations of the four hands). This yields roughly $3.548 \cdot 10^{25}$.