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Counting

Permutations

Let us examine a real-life problem. As a university professor I feel that teaching twenty students for one hour a week is too much of a demand on my time and that it cuts into my research (on sour mash). Therefore, in order to cut back on my teaching load in future classes, I have decided to flunk most of my students this semester. Specifically, I am going to give one A, one B, one C, one D, and I'll flunk everyone else. Also, in order to keep from taxing my mental resources, I will pick the four passing students randomly.

The question of this section is *how many ways can I pick the four students out of twenty*? In particular, the order of students matters. If I give Fred an A, Mary a B, Stanley a C, and Repunzel a D, that is different than giving Mary an A, Fred a B, Stanley a C, and Repunzel a D. In other words, *how many ways can I pick four out of twenty students* ***with*** *respect to order*? The quantity I am asking for is *the number of permutations of four objects out of twenty* and it is denoted by *P(20, 4).* On a calculator the number of permutations of r objects out of n will usually be denoted by $P_{n,r}$ or by ${}_nP_r$. To answer, our specific problem of choosing P(20,4) is quite easy. There are 20 possible choices for who gets an A. Having chosen the A there are 19 remaining choices for the B. That leaves 18 choices for the C, and 17 choices for the D. Consequently, the number of possibilities is 20·19·18·17 = 116,280.

The formula for P(n, r) is derived exactly as the above problem was solved:

$$\mathrm{P}(n,r) \quad = \quad n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+2)\cdot(n-r+1)$$

Formula 1 A Good Computational Form for Permutations

Notice that the expression n(n–1)(n–2)·...·(n–r+1) has r terms. For example, to choose 3 out of 5 objects *with* respect to order, there are 5·4·3 = 60 possibilities. This formula can be computed using an iteration from 1 to r (or 0 to r–1). A general formula for P(n, r) that you will see in many texts is:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Formula 2 A Good Formula for Proving Theorems About Permutations

Again, n! is read *n factorial* and means n·(n−1)·(n−2)·...·3·2·1 (with 0! defined as 1). **Formula 2** works through cancellation: The terms in the denominator cancel terms in the numerator to give exactly the same expression as in **Formula 1**. To see why **Formula 2** is bad computationally consider P(100,2). Lastly, a recursive formula for P(n, r) which is excellent computationally is:

$$P(n,1) = n$$
$$P(n,r) = n \cdot P(n-1,r-1), \quad r > 1$$

Formula 3 A Recursive Formula for Permutations Which is Excellent for Computations

The foundation of **Formula 3** is that there are n ways to pick 1 out of n objects; and the number of ways to choose r out of n objects is n, the number of ways of choosing the first object, times the number of ways of choosing r-1 objects out of the remaining n-1 objects.

Suppose we have n objects to line up, and we want to know how many ways to line them up with order respected. This is a problem you probably learned to solve in high school. There are n choices for the first object; n–1 choices for the second object, and so on, down to 1 choice for the last object. Multiplying these numbers we get n! which happens to be equal to P(n,n): the number of ways of permuting n out of n objects.

☐ **Exercise 1** Suppose we want to arrange n objects in a circle (let's assume there are exactly n positions). We are interested in counting the number of ways of arranging the objects with respect to order. What is important is not where an object sits, but its relationship to the other objects. How many ways can we do this? Hint: The answer is not n!.

☐ **Exercise 2** Show the equivalence of the three formulas for P(n, r).

To prove three objects are equivalent, such as objects x, y, z, it is not necessary to directly prove each object implies each other object. That would mean six cases to be solved. It is sufficient to prove, for example, that x implies y and y implies z and z implies x. In this way you have solved only three implications, but x implies z follows from x implies y and y implies z. z implies y follows from z implies x and x implies y. y implies x follows from y implies z and z implies x.

Binomial Coefficients

Let us apply the above formula to count poker hands. If we compute the number of ways of choosing 5 out 52 cards, P(52, 5), we get 311,875,200. However, this is not right. P(n, r) counts the order of cards chosen. For example, under this formula the hand A♥9♦8♥5♣2♠ is counted separately from 5♣2♠A♥8♥9♦, because though the cards are the same, they are ordered differently. However, in poker we don't care what order the cards are dealt in. Any five-card hand like A♥9♦8♥5♣2♠ has P(5,5) = 5! = 120 permutations (or orders). Hence in the formula P(52,5) each hand is counted 120 times. The correct number of poker hands is $\frac{P(52, 5)}{5!}$ which is 2,598,960. Whereas we refer to objects chosen with respect to order as permutations, objects chosen without respect to order are called *combinations* and are denoted C(n, r) or $\binom{n}{r}$. The latter notation is what you will usually see in books. On calculators you will see $C_{n,r}$ or ${}_nC_r$. We have essentially derived the formula:

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!}$$

Formula 4 The Number of Combinations of r Out of n Objects

C(n, r) is also known as a *binomial coefficient* for a reason to be explained later. The binomial coefficient C(n, r) is the number of way of picking r out of n objects without respect to order. Another way of looking at it is that C(n, r) is the number of ways of arranging n objects where r are distinguished by some characteristic. In terms of our problem at the beginning of the chapter of picking 4 students out of 20, originally their order mattered because the four students were to get different grades. Suppose now that we are picking 4 students to be shot, presumably still with the intent of lowering my future enrollments (I've been to more than one school that would have approved of this). Now, if we were to pick Mary, Fred, Stanley, and Repunzel, it would not matter in what order we picked them. Therefore, the correct answer is our original answer 116,280 divided by 4! and giving C(20,4) = 4,845.

Most texts give the following formula:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Formula 5 A Standard Formula for Combinations Which is Not a Good Computational Formula

This definition is lousy computationally (again consider C(100,2)) but it is useful for proving formulas such as:

$$\binom{n}{r} = \binom{n}{n-r}$$

Formula 6 A Useful Identity

An easy way to see that **Formula 6** is true, is to consider that selecting r out of n objects to use for some purpose is the same as selecting the n - r objects that you will not use.[1] A good computational formula is:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-r+2)(n-r+1)}{r(r-1)(r-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}, \quad r > 0; \quad \binom{n}{0} = 1$$

Formula 7 Another Formula for Combinations

It is useful to remember that there are r terms in both the numerator and denominator of **Formula 7** . Before using **Formula 7** you should use **Formula 6** to minimize r.

A computationally excellent recursive formula for combinations is:

[1]Note this logic does not apply to permutations. It is worthwhile to think about this.

$$C(n,0) = 1$$

$$C(n,r) = \frac{n}{r}C(n-1,r-1), \quad r > 0$$

Formula 8 A Computationally Good Recursive Formula for Combinations

This formula can be derived immediately by using **Formula 3** . Again **Formula 6** should be used to reduce r before using **Formula 8** . **Formula 8** is virtually identical to **Formula 7** but has one advantage. **Formula 7** should be done using a iteration scheme and such that the operations are done in the order: $\left(\frac{n}{r}\right)\left(\frac{n-1}{r-1}\right)\left(\frac{n-2}{r-2}\right)\cdots\left(\frac{n-r+1}{1}\right)$. If instead we multiply out the numerator first, and then we multiply out the denominator, we are liable to encounter overflow. With **Formula 8** , the operations are done in the correct order automatically.

☐ **Exercise 3** Prove that **Formula 5** , **Formula 7** , and **Formula 8** are equivalent.

The Binomial Graph[1]

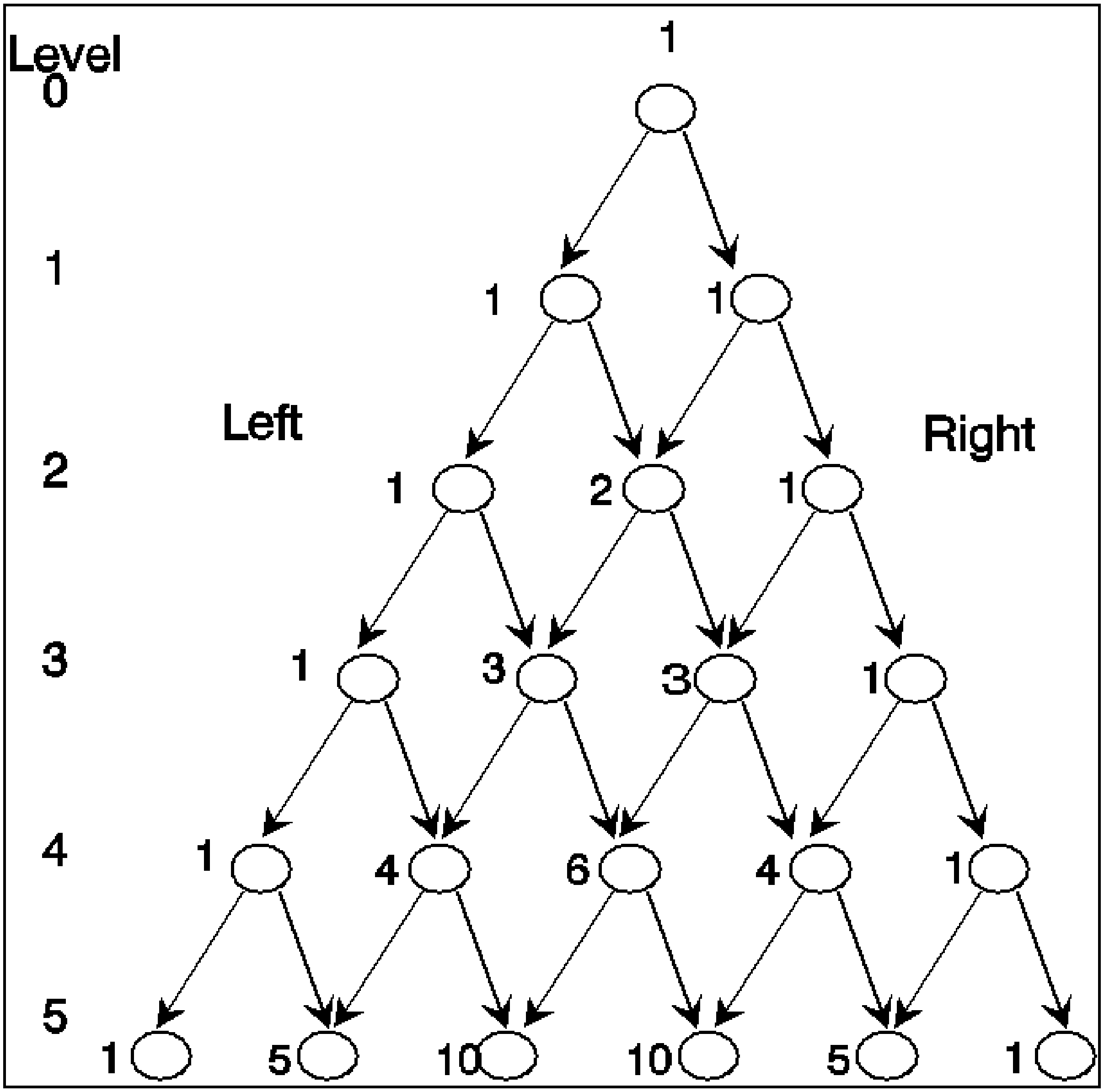

Figure 1 Pascal's Triangle (as a Road Map)

Consider the road map in **Figure 1**. Think of each vertex as an intersection. We start at the intersection on level 0. Our question is *how many ways are there for us to reach a given*

[1]This section is inspired by the great George Pólya's lecture at the beginning of *Notes on Introductory Combinatorics* by Pólya, Tarjan, and Woods. Birkhäuser, 1983.

intersection? Notice that each intersection has a level. At every intersection we come to, we can make one of two choices. We can go left or right. If we are going to the 5'th level, we must make 5 choices. If we choose to make 3 left turns, it doesn't matter at which intersections that we turn left; we will come to a unique vertex which we can denote by [5,3] because it is on level 5 and it takes 3 left turns (and 2 right turns) to get there. This last statement is the heart of the argument, so study **Figure 1** until you are sure that the statement is correct. For the general intersection [n, r] n denotes the level, and r is between 0 and n, because to reach an intersection on level n, we have n decisions to make and we can choose to go left anywhere from 0 to n times. We can look at each intersection on level n in terms of the n left-right decisions it took to get there. Each intersection is a list of n symbols, some of them L and some of them R. But since all that matters is the number of L symbols, the number of ways to get to intersection [n, r] is just $\binom{n}{r}$. Notice that the graph illustrates what we already know: $\binom{n}{0} = \binom{n}{n} = 1$. Now consider any intersection in the interior of the graph. To get to that intersection we must go through the intersection on the previous level immediately to the left or the intersection on the previous level immediately to the right. The number of ways to get to the new intersection is the sums of the numbers of ways to get to the two intersections immediately above it. This is illustrated in **Figure 1** where each intersection [n, r] is labeled by $\binom{n}{r}$. The figure is essentially Pascal's triangle and we have essentially proven the famous identity:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}; \quad n,\ r \geq 1$$

Formula 9 Pascal's Binomial Identity

This identity can also be understood in another way. Suppose we want to choose r out of n objects without regard to order. We mark one of the n objects with an X. Then the number of ways to choose r out of the n objects is the number of ways to choose r objects not including the marked object plus the number of ways to choose the r objects including the marked object. In the former case we are choosing r out of n - 1 objects and in the latter case, we choose the marked object and we have to choose the remaining r - 1 objects out of n - 1 remaining objects. There are many identities (known as *binomial identities*) like **Formula 9** . Many of them can be understood by appealing to **Formula 1** . **Formula 9** is also the basis for a recursive algorithm for generating binomial combination numbers. This algorithm has two terminating conditions: C(n, n) = 1, and C(n, 0) = 1. The recursive algorithm based upon **Formula 9** is very inefficient in both time and memory. The reason is that the algorithm does many of the calls repeatedly. For example to compute C(5,3) by **Formula 9** you must compute C(4,3) and C(4,2), but to computing C(4,3) and C(4,2) each requires you to compute C(3,2).

☐ **Exercise 4** Study the inefficiency of computing C(5,3) recursively using **Formula 9** and C(n,n) = 1, and C(n,0) = 1. Study this by constructing a tree. The top vertex would be C(5,3) and right below it would be vertices corresponding to C(4,3) and C(4,2) since they are called by C(5,3). Continue until each line of descent ends in a terminating condition.

☐ **Exercise 5** Write a recursive algorithm for computing $\binom{n}{r}$ based upon **Formula 9** .

The Binomial Expansion

An expression like X + Y is called a *binomial*. If we raise the binomial to an integer power such as 5, we get terms such as XY^4, X^2Y^3, and Y^5. The exponents on each product must add to the power, in this case 5. The only problem is to compute the constants that we multiply against each term. One way of doing this is to simply read off the level of Pascal's triangle corresponding to the power. For example to compute $(X + Y)^5$, we read off the fifth level of Pascal's triangle to get:

$$(X + Y)^5 = 1\cdot X^5 + 5\cdot X^4\cdot Y + 10\cdot X^3\cdot Y^2 + 10\cdot X^2\cdot Y^3 + 5\cdot X\cdot Y^4 + 1\cdot Y^5$$

This is why the combination numbers $\binom{n}{r}$ are called *binomial coefficients*. We will not use the binomial theorem, but if you intend to continue in the mathematical sciences you need to memorize it.

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

The Binomial Theorem

<u>Example</u> The sum $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$ is the number of ways of picking subsets out of n objects. First there is the empty subset of which there is 1, then there is the number of subsets of size 1, $\binom{n}{1} = n$, and so on. Another way to count the number of subsets of n objects is to consider that each element has two possibilities: either the element is in the subset or it is not. The number of subsets is then 2×2×2×...×2 or 2^n. Therefore we have:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Probability Problems With Counting

This book will begin the study of probability in section 20. However, we now have the tools to begin the study of basic probability problems. For example, to figure out the probability of being dealt a particular poker hand, you divide the number of poker hands of the type you are interested in by the total number of poker hands which is: $\binom{52}{5} = 2{,}598{,}960.$ If you don't know the various poker hands, ask someone; there are not many and they are easy to learn. In fact of all the standard hands (pair, two-pair, three-of-a-kind, straight, flush, full house, four-of-a-kind, straight-flush) probably the most difficult to calculate is the two-pair hand. The probability of being dealt two-pair is calculated as follows:

Example We need to calculate the number of two-pair hands that can be dealt by an ordinary 52 card deck. First we calculate the number of ways we can choose 2 values out of 13. This is where many people go wrong. They calculate 13 choices for the first value and 12 for the second value, giving a total of $13 \cdot 12 = 156$ possible cases. However, this is wrong because it counts aces and kings, and kings and aces as two different cases. Therefore it is off by a factor of 2. What we want is $\binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = 78.$ Now for each value (such as Aces and Kings) we need to pick 2 out of 4. This can be done in

$\binom{4}{2} = \frac{4\cdot 3}{2\cdot 1} = 6$ ways, and has to be done for each value. This means we have to pick the fifth and remaining card. Now there are 4 cards already selected and which can't be selected for the fifth card. Also, there are 4 cards remaining in the two values of the pairs. They can't be selected either because that would give us a full house. Therefore the remaining card must be selected out of 44 cards and that can be done in $\binom{44}{1} = \frac{44}{1} = 44$ ways. The total number of two-pair hands we can be dealt is thus: $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1} = 123{,}552.$

The probability that you are dealt a two-pair hand is then:

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} = \frac{123{,}552}{2{,}598{,}960} = .047539 \ .$$

The problems that immediately follow depend on proficiency in counting and with binomial coefficients. Solving these problems is an art that one learns with practice. It is a good example of why the answers to **all** the problems are in the back of the book.

☐ **Exercise 6** Calculate the probability in poker of being dealt a one-pair hand.

☐ **Exercise 7** Calculate the probability of being dealt a three-of-a-kind hand.

☐ **Exercise 8** Calculate the probability of being dealt four-of-a-kind.

☐ **Exercise 9** Calculate the probability of being dealt a full house.

☐ **Exercise 10** Calculate the probability of being dealt a straight flush.

☐ **Exercise 11** Calculate the probability in poker of being dealt a flush.

☐ **Exercise 12** Calculate the probability of being dealt a straight.

☐ **Exercise 13** Suppose you are playing with a deck that has two jokers as wild[1] cards. What is the probability of being dealt two pair?

☐ **Exercise 14** Suppose you are playing with an ordinary 52 card deck, and suppose one-eyed jacks (there are two) are wild. What is the probability of being dealt two pair?

☐ **Exercise 15** Suppose you are playing with an ordinary 52 card deck, and suppose one-eyed jacks are wild. What is the probability of being dealt three-of-a-kind?

☐ **Exercise 16** The Florida Saturday lottery requires picking 6 out of 49 numbers for a jackpot typically around 7 million dollars. If you choose 5 out of the six numbers correctly, you can make around 5 thousand dollars. If you buy one ticket (that is you choose 6 numbers) what is the probability that you get exactly 5 out of the 6 correct numbers?

[1] I use problems with wild cards for mathematical purposes. Inclusion of such problems does not mean that I condone playing poker with wild cards.

1. It matters not where we place the first object (or which is the first object). The problem is then where do we place the other objects relative to the first? There are $n-1$ candidates for the first seat on the left of the first object. There are $n-2$ candidates for the seat immediately to the left of that object, etc. Altogether there are $(n-1)!$ seating arrangements.

2. The first formula is equivalent to the second formula can be seen by canceling the denominator with the numerator in the second formula. In order to finish the problem, it is sufficient to prove the first formula equivalent with the third. Let us consider the first formula when $r \rangle 1$: $P(n, r) = n(n-1)...(n-r-1)$. Notice that the part $(n-1)...(n-r-1)$ is, by the first formula, $P(n-1,r-1)$ implying $P(n, r) = nP(n-1,r-1)$. If $r = 1$ both formulas give the same result. Clearly, then the first formula implies the third. Going the other way, we repeatedly apply the third formula. At each step we decrease r by 1. Eventually r will equal 1 and we will be finished. Applying these steps it can be seen we achieve the same result as the first formula. (The more sophisticated among you may suspect that the best way to show the first and third formulas are equivalent is by proof by induction. You may be right. Induction is extremely close to this topic of recursion. However, this edition of this text does not cover induction. I may put it in a future edition.)

3. That **Formula 5** and **Formula 7** are equivalent can be seen by canceling the $(n-r)!$ out of the numerator and denominator of **Formula 5** . To finish the problem, it is sufficient to prove **Formula 7** and **Formula 8** are equivalent. They are clearly equivalent when $r = 0$. If $r \rangle 0$ factor n/r out of **Formula 7** and apply **Formula 7** to the remainder and you get the second expression of **Formula 8** . This shows that the two formulas are consistent with each other. To see that **Formula 8** terminates, notice that each step (of the second part) r is decreased by 1. Eventually, $r = 0$ and the algorithm is finished.

4. You should get a tree with a whole lot of vertices.

5. Again, this particular recursion is quite inefficient as you should discover from the previous exercise. The algorithm goes like this:

```
Define Function C(n, r)
Begin
        If n < r or r < 0 then "error"
        If r = 0 Then Return 1
                Else Return C(n−1, r) + C(n−1, r−1)
End
```

The last statement in the algorithm makes two recursive *calls* on itself. This can easily be implemented in a language such as Pascal or Microsoft's QuickBasic. Again, it will be highly inefficient whereas the recursive algorithm based upon **Formula 8** is highly efficient.

6. There are C(13,1) = 13 ways to pick the value of the pair. There are C(4,2) = 6 to pick 2 out of the four cards. There are C(12,3) = 220 ways to pick the remaining 3 values. For each of these there are C(4,1) = 4 ways of picking the particular card. The probability of being dealt a one-pair hand is 13·6·220·4·4·4 divided by the total number of poker hands, 2,598,960. This yields .422569 (roughly).

7. There are C(13,1) = 13 ways to pick the value of the three-of-a-kind and there are C(4,3) = 4 ways to pick the three cards. There are C(12,2) = 66 ways of picking the other two values and there are C(4,1) ways of picking each of those cards. The total number of three-of-a-kind hands is 13·4·66·4·4 = 54,912 with a probability of .021128 (roughly).

8. The number of ways of picking the value of the four-of-a-kind is C(13,1) = 13. The number ways of picking the particular 4 cards is C(4,4) = 1 (I included this for consistency with the previous two cards). The number of ways of picking the remaining card is C(48,1) = 48 (if we found this like in the previous two problems it would be C(12,1)·C(4,1) = 48). The number of 4-of-a-kind hands is 13·48 = 624 for a probability of .000240 (roughly).

9. There are C(13,1) = 13 ways to pick the 3-of-a-kind and there are C(12,1) = 12 ways to pick the two-of-a-kind. (Note, it is not correct to say there are C(13,2) ways to pick the two values; the reason is that in this case, order counts!) There are C(4,3) = 4 ways to pick the 3-of-a-kind (once their value is chosen) and there are C(4,2) = 6 ways to pick the 2-of-a-kind. The number of full house hands is 13·4·12·6 = 3744 for a probability of .001441 (roughly).

10. A straight-flush can be characterized by two things: its suit and its highest card. There are C(4,1) = 4 ways of picking the suit. The high card ranges from ace down to 5 (in that last case, the ace is the lowest card) for a total of 10 high cards within the suit. Altogether there are 40 straight-flushes with a probability of .000015 (roughly). (Compare this to a total of 48 hands with 4 aces.)

11. There are C(4,1) = 4 ways of choosing the suit. Within the suit there are C(13,5) = 1287 of picking the 5 cards. Hence there are 4·1247 = 5148 flushes. However, the 5148 includes the 40 straight-flushes which we must count separately. Hence there are really 5108 flushes for a probability of .001965 (roughly).

12. A straight is characterized by its top card which can be anything from a 5 to an ace. That gives us 40 possible top cards. Each of the remaining cards is known except for the suits, for which each has C(4,1) = 4 possibilities. The total number of straights is then 40·4·4·4·4 = 10240. Again, we must subtract the straight-flushes (40) for a total number

of flushes 10200, and a probability of .003925 (roughly).

13. If you get two-pair using wildcards, give up this book and this course; give up college. Above all else, don't play poker. Any two-pair hand with wild-cards could be better cast as at least 3-of-a-kind. Hence the total number of two-pair hands is exactly the same as in a 52-card deck as worked out in the example above, 123,552. However, the number of poker hands is now C(54,5) = 3,162,510. The probability of being dealt 2-pair is .039068.

14. Before studying this problem, you might want to review the two-pair example in the text. Again a wild-card should not appear in a two-pair hand. Let us first solve the problem for the number of two-pair hands when no pair is jacks. There are C(12,2) = 66 ways to pick the values of the pairs. There are C(4,2) = 6 ways of picking each of the pairs (once the values are chosen). There are 42 cards to pick the remaining card from (this is because the remaining cards of the two values are excluded as well as the two wild jacks). There are so far 66·6·6·42 = 99,792 two-pair hands. Let us now count the number of two-pair hands where one pair is jacks. There is essentially only one way we can have two jacks (when there is another pair!) and that is the one pair of two-eyed jacks. There are remaining C(12,1) = 12 values for the remaining pair and out of that we can choose C(4,2) =6 pairs. There are 44 cards to pick the remaining card from. Hence the number of two-pair hands is 99,792 + 12·6·44 = 102,960. The probability of two pairs is then .039616 (roughly). Hence having two wild cards out of the 52 card deck lowers the probability of being dealt two pair.

15. (This is a fairly difficult problem; only a sadistic teacher would expect you to get it right.) Let us consider hands without wild-cards, there are C(12,1) = 12 ways to pick the value of the 3-of-a-kind. There are C(4,3) = 4 ways to pick the 3. There are C(11,2)·C(4,1)·C(4,1) + C(11,1)·C(4,1)·C(2,1) = 968 ways of picking the remaining two cards (one may be a two-eyed jack). Let us consider hands with one wild-card. There are C(2,1) = 2 ways of picking the wild-card. There are C(12,1)·C(4,2) = 72 of picking the remaining part of the 3-of-kind if not jacks. That leaves C(11,2)·C(4,1)·C(4,1) + C(11,1)·C(4,1)·C(2,1) = 968 of picking the last two cards . If the three of a kind are jacks, then the other two jacks are the two-eyed jacks and there are C(12,2)·C(4,1)·C(4,1) = 1056 ways to pick the remaining two cards. Let us consider hands with two wild cards. The wild cards are uniquely determined. If the three-of-a-kind are not jacks, there are C(12,3)·C(4,1)·C(4,1)·C(4,1) +C(2,1)·C(9,2)·C(4,1)·C(4,1) = 15232 of picking the other three. If the three-of-a-kind are jacks, there are C(2,1) = 2 ways to pick the two-eyed jack. The remaining two cards must be less than a jack (otherwise the three-of-a-kind would be a higher value). Hence there are C(9,2)·C(4,1)·C(4,1) = 576 ways to pick the other two cards. The total number of hands is 12·4·968 + 2·72·968 + 2·1056 + 15232 + 2·576 = 190,272. The probability of 3-of-a-kind is .07321 (roughly). Notice that the wild-cards decreases the probability of 2-pair and greatly increases the probability of 3-of-a-kind.

16. There are C(49,6) = 13,983,816 possible lottery outcomes. The number of ways of getting exactly 5 of the correct numbers is C(6,5)·C(43,1) = 258. The probability is then .0000184499 (roughly).